Milos Savic

WHERE IS THE LOGIC IN PROOFS?

Why is this an interesting question?

- Often university mathematics departments teach some formal logic early in a transition-to-proof course in preparation for teaching undergraduate students to construct proofs.

- There are some that believe that formal logic should be taught first, separately (Epp, 2003) and some that believe that logic need not be explicitly taught at all (Hanna & de Villiers, 2008).

Why is this an interesting question? (cont.)

- The aim of the analysis is to find the logic in proofs so that the question of how it should be taught can be better understood.

- If formal logic occurs quite a bit, then teaching a unit on predicate and propositional calculus might be a good idea; if formal logic is infrequent, then teaching logic in context, while teaching proving, might be more effective.

The setting

- The proofs were the whole output of the 9 students of an "Understanding and Constructing Proofs" course at a large Southwestern University. This course was for advanced undergraduates and beginning graduates.

- Students constructed proofs at home and presented on the blackboard and discussed

- All 42 proofs were then approved by the professor and copies were given to everyone in the class.

- Theorems include sets, functions, real analysis, algebra, and topology.

The coding

- In this study, I coded 42 student-constructed proofs of theorems using a "chunk-by-chunk" analysis.
- There were several iterations of the coding process during which the categories of chunks emerged.
- One iteration included having two mathematics professors coding several theorems and meeting twice to discuss the coding.

Chunk-by-chunk Analysis

- The "chunks" are similar to those used in analyzing short-term memory (Miller, 1956). They are small phrases that can be taken together as a "meaningful unit" in thinking.
- Some chunks can be sentences, others can be one or a couple of words, but they are always meant to refer to a moment or unit in the proof.
- The two professors mentioned earlier were in agreement on over 80% of the chunks in 4 proofs during one chunking iteration.

Chunk-by-chunk Analysis (cont.)

Theorem 3: For sets $A, B,$ and C, if $A \subseteq B$ then $C - B \subseteq C - A$.

Proof: Let A, B, and C be sets such that $A \subseteq B$. Suppose $x \in C - B$. Then $x \in C$ and $x \notin B$. By $A \subseteq B$ we have $x \notin A$; hence $x \in C - A$. Therefore, $C - B \subseteq C - A$.

1. Let A, B, and C be sets
2. such that $A \subseteq B$.
3. Suppose $x \in C - B$.
4. Then $x \in C$ and $x \notin B$.
5. By $A \subseteq B$
6. we have $x \notin A$;
7. hence $x \in C - A$.
8. Therefore, $C - B \subseteq C - A$.

The categories

- During the coding of the chunks, 13 categories emerged.
- I will describe five of the categories; two about logic and the three that occurred most often.

5 categories

- Informal Inference (common sense) (**II**)
  - Can be made without bringing to mind formal logic (by students at the beginning of a transition-to-proof course)
  - A common example is modus ponens
- Formal Logic (**FL**)
  - Inference requiring predicate and propositional calculus of the kind taught in a transition-to-proof course
  - Beginning transition-to-proof students might not know this "formal logic"
  - An example would be: if $x \notin B \cup C$, then $x \notin B$ and $x \notin C$

5 categories (cont.)

- Definition (**DEF**)
  - The chunk is immediately derived from the definition
- Assumption (**A**)
  - Introducing a mathematical object or assuming properties of the object
  - Two Sub-categories
    - Example: For the theorem "For all $n \in \mathbb{N}$, if $n > 5$ then $n^2 > 25$."
    - Choice: "Let $n \in \mathbb{N}$"
    - Hypothesis: "Suppose $n > 5$"
- Interior Reference (**IR**)
  - Referring to a chunk or chunks stated earlier in the proof

Example 1

- Theorem 2: For sets $A, B,$ and C, if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

- Proof: Let $A, B,$ and C be sets. Suppose $A \subseteq B$ and $A \subseteq C$. Suppose $x \in A$. Then $x \in B$ and $x \in C$. That means by the def of intersection $x \in B \cap C$. Therefore, $A \subseteq B \cap C$.

Example 1 (cont.)

Let A, B, **and** C **be sets.**	Assumption (Choice)
Suppose $A \subseteq B$ **and** $A \subseteq C$**.**	Assumption (Hypothesis)
Suppose $x \in A$**.**	Assumption (Hypothesis)
Then $x \in B$	Informal inference
and $x \in C$**.**	Informal inference
That means by the def of intersection $x \in B \cap C$**.**	Definition of intersection
Therefore, $A \subseteq B \cap C$**.**	Conclusion statement/Definition of subset

Example 2

- Theorem 38: If X is a Hausdorff space and $x \in X$, then $\{x\}$ is closed.

- Proof: Let X be a Hausdorff space. Let $x \in X$. Note $\{x\} = X - (X - \{x\})$. Suppose $y \in X$ and $y \neq x$. Because X is Hausdorff, there is an open set P_y for which $y \in P_y$. There is also an open set R_y such that $x \in R_y$ and $P_y \cap R_y = \emptyset$. Suppose $P_y \nsubseteq X - \{x\}$, then $x \in P_y$, but $x \in R_y$. Therefore $x \in P_y \cap R_y$, which is a contradiction. Therefore, $P_y \subseteq X - \{x\}$. Thus for every $y \neq x$ there is an open set P_y where $y \in P_y$ and $P_y \subseteq X - \{x\}$. The union of all P_y is equal to $X - \{x\}$, which is thus an open set. Therefore $\{x\}$ is closed, being the complement of an open set.

Example 2 (cont.)

Let X be a Hausdorff space.	Assumption (Hypothesis)
Let $x \in X$.	Assumption (Hypothesis)
Note $\{x\} = X - (X - \{x\})$.	Formal Logic
Suppose $y \in X$ and $y \neq x$.	Assumption (Choice)
Because X is Hausdorff,	Interior reference
there is an open set P_y for which $y \in P_y$. There is also an open set R_y such that $x \in R_y$ and $P_y \cap R_y = \emptyset$.	Definition of Hausdorff
Suppose $P_y \nsubseteq X - \{x\}$,	Assumption (Hypothesis)
then $x \in P_y$,	Informal inference
but $x \in R_y$.	Interior reference
Therefore $x \in P_y \cap R_y$,	Definition of intersection
which is a contradiction.	Contradiction statement

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Results

- In the 42 proofs, consisting of 673 chunks, formal logic (**FL**) constituted 1.9% of the chunks (13 chunks), while informal inference (**II**) was 6.5% (or 44 chunks).
- Definition (**DEF**): 30% of the proof chunks (or 203 chunks)
- Assumption (**A**): 25% of the proof chunks (166 chunks)
- Interior reference (**IR**): 16% of the proof chunks (108 chunks)

Why such a small percentage of formal logic?

- The course intended to cover a wide variety of kinds of proofs, causing many of the proofs to be based mainly on definitions.
- The coding did not consider the *implicit* logical actions in the proving process or in the structuring of proofs.

Homology Class

- In Fall 2010, I took a Homology course and chose to code 10 proofs using the same categories.
- The proofs were from another student in the class who got a perfect score on all homework.
- I found that less than 1% of the 170 chunks were coded as formal logic (**FL**), while informal inference (**II**) had 10%.
- Definition (**DEF**), assumption (**A**), and interior reference (**IR**) were the three highest percentages (21% vs. 30%, 18% vs. 24% and 17% vs. 16% respectively).

Logic-like structures

- A logic-like structure preserves truth value in an argument, yet is not in the language of predicate and propositional calculus.
- For example, if you see a situation where the theorem states "For all $x \in A$, $P(x)$", one starts with "Let $x \in A$." and reasons to "$P(x)$"
- Another example would be to prove there is a unique x so that $P(x)$, one starts with "Suppose $P(a)$ and $P(b)$" and reasons to "$a = b$". This logic-like structure shows up in proving that an identity is unique in a semigroup.

Examining students' approach to proving

- I videoed and interviewed 3 students that from the class one year later.
- 45 min. were focused on the uninterrupted, think-aloud production of the proof, followed by 15 min of interview.
- One page of notes was given to the students starting with the definition of semigroup and supplying all information needed to prove the theorem.
- The theorem: Every semigroup has at most one minimal ideal.

Results

- No one finished the proof correctly after 45 min. One student finished, but with some gaps in her proof.
- Every student immediately considered a semigroup S, and all approached the proof by assuming two, or n, minimal ideals.
- After this, each student proved the theorem differently, but that did not mean more logic was used.

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Future research

- I would like to code the structures of proofs, or as Solow (1982) calls it, "Proof techniques". These same techniques include many of the logic-like structures mentioned earlier.
- I would like to expand on how behavioral knowledge of logic-like structures help to reduce the burden on working memory.
- Lastly, I've had several professors ask me to code the proofs in a chapter of a textbook to see how much formal logic occurs in the proofs.

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Questions? Comments?

- Please email me with any other thoughts or suggestions: milos@nmsu.edu