Experiencing Mathematics: From Middle School to Mathematicians

Milos Savic

Michigan State University

May 15, 2013

Outline of the talk

- Math Snacks
  - How we create the games and animations
  - Math camps for both teachers and students
- Observing mathematicians proving alone
  - New data collection technique
  - Impasses – “getting stuck”
  - Incubation
- Future research

Math Snacks

- What is a “math snack”?
  - Small (3-10 minutes) animation or engaging game
- Other gaps in learning mathematics
  - Key concepts that have “Vertical power”
- Concept Image vs. Concept Definition (Tall & Vinner, 1981)
  - Multiple visual representations

NSF Funded Grant

The Existing Knowledge Gaps

- An analysis of over 24,000 NM student scores from the NMSBA in 3rd -8th grade in high-needs NM districts showed consistent areas of weakness, which we call “gaps.”
- 3 Criteria were used when selecting a gap:
  - Is it a key concept in Mathematics?
  - Is it easy to build on?
  - Can technology make the concept more accessible?

Learning Goals

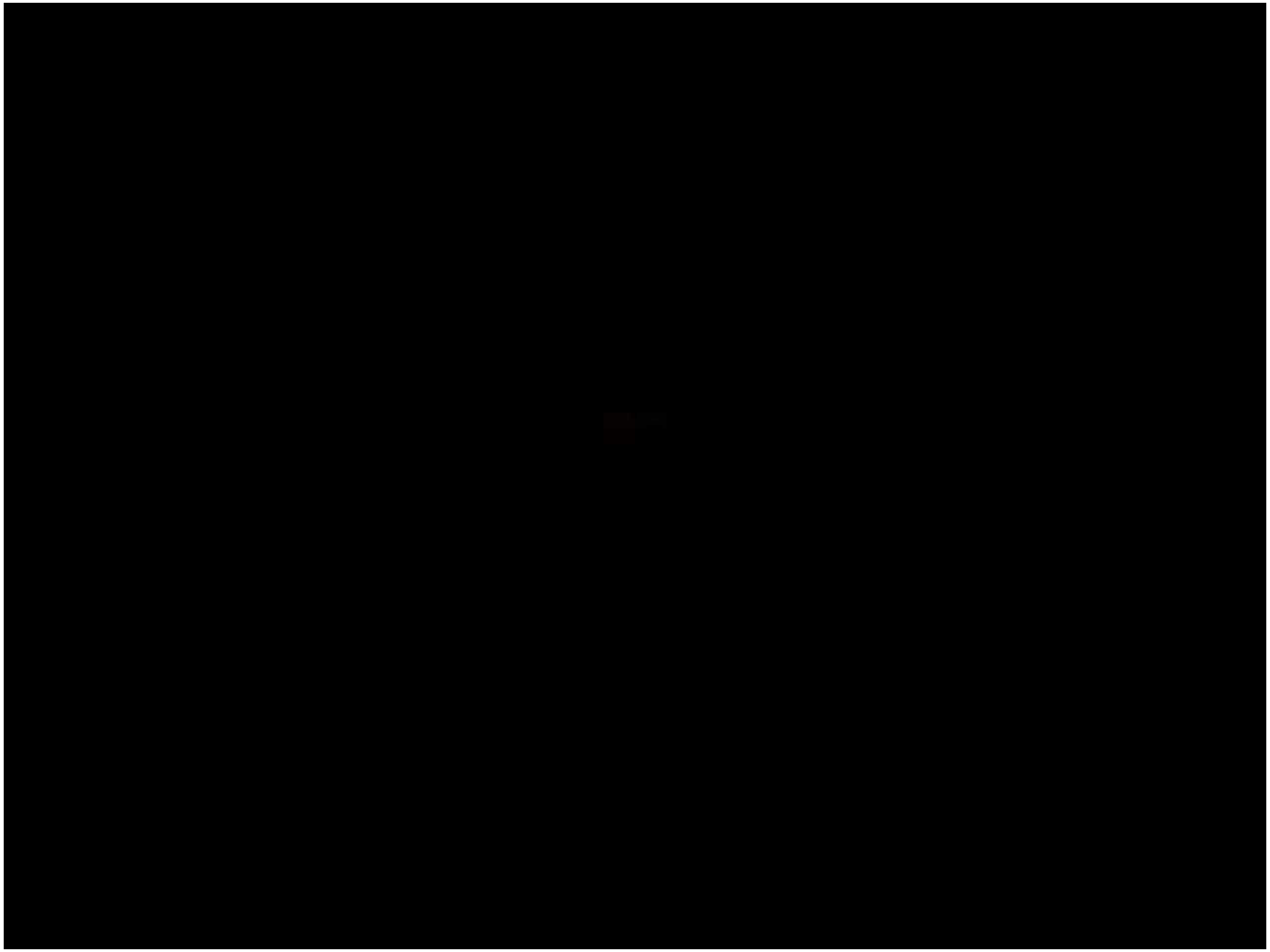
- Developed based on criteria from previous slide
  - Key concept, easy to build, and technologically accessible
- The goals have 3 broad mathematical “big ideas” (Ratio and Proportion, Number Sense with a Number Line, Coordinate Plane), with examples and scenarios in real life

Product: Math Snacks

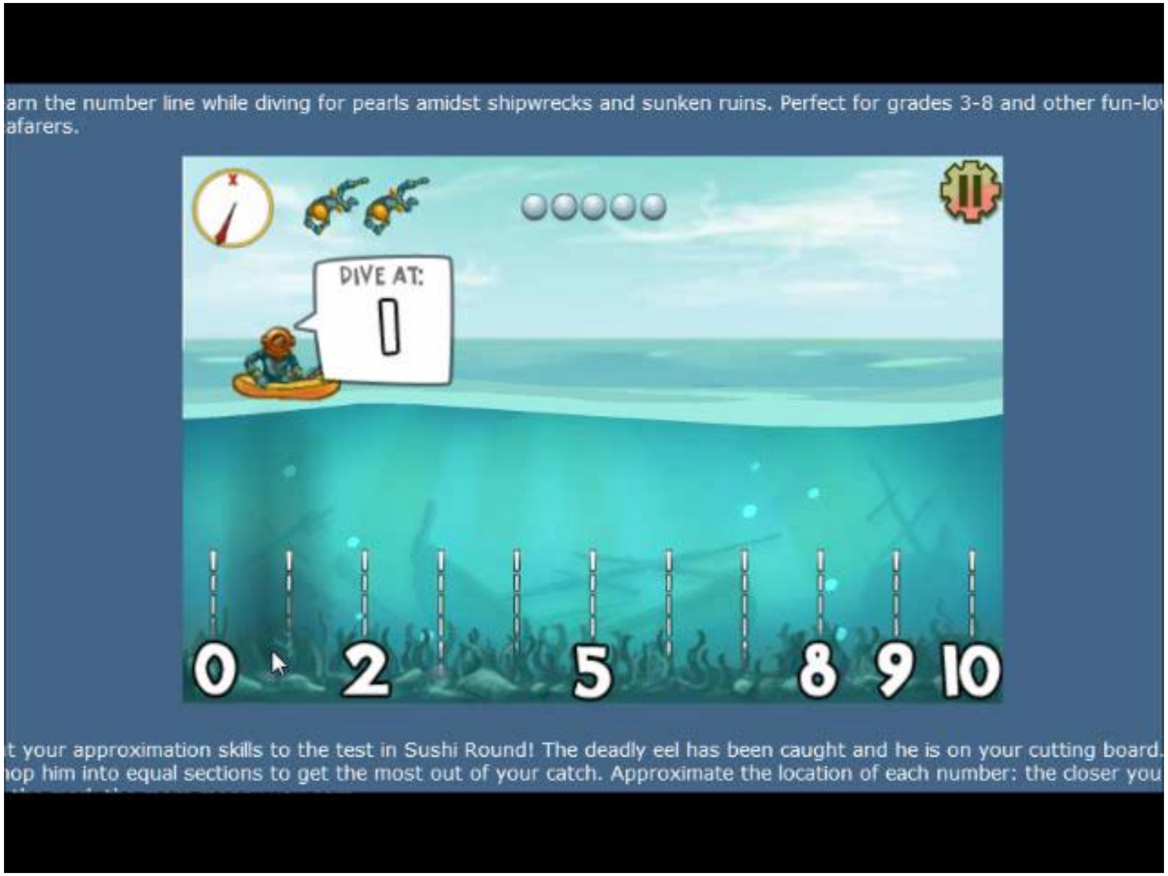
- Was created with the student's concept image in mind
- Has gone through many iterations
- Was pilot tested with both students and teachers in the NMSU Learning Games Lab

Examples

- Example Gap: Using mathematical models to interpret and analyze numerical data
- Example Goal: Use a number line to order, compare, and measure with whole numbers, fractions, mixed numbers, and decimals.
- Example products: “Number Rights” and “Pearl Diver”



Game: Pearl Diver



Technology

- Available online at www.mathsnacks.com
  - Eleven Math Snacks: Number Rights, Overruled!, Scale-ella, Bad Date, Atlantean Dodgeball, Ratey the Math cat, Monster School Bus, Pearl Diver, Game Over Gopher, Gate, and Ratio Rumble
- iPad, iPhone, and iPod distribution
- Video lessons with animations available
- Learner and Teacher guides also available
- Cost: Free

Math Games Camp

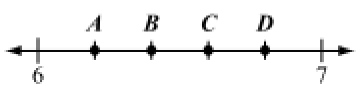
- Two Summer Camps
  - Two teachers per 8-12 students, 8-12 teachers total, 40-60 students total
  - The teachers could select one to three math snacks for use in their classroom
  - Three meetings with the teachers before the camp, with debriefing sessions after each day’s training
  - 4 day camp, 8 AM-12 PM each day for the students, teachers stayed 1 hour extra for debrief
  - Overarching project for the camp with each group

Math Games Camp

- Fall 2011 and Spring 2012
  - 3 teachers with their students
  - 3 consecutive Tuesdays in Fall and Spring, 3 consecutive Wednesdays in Spring
  - Focus on professional development
    - First day: The Math Snacks team (Dr. Karen Trujillo and I) taught the lesson
    - Second day: Half Math Snacks team, Half teacher with assistance
    - Third day: The teacher taught the lesson
    - In between days, the math snacks team and the teachers would meet to plan and debrief

Pilot Testing

- The Math Snacks team tried a pilot test out in Fall 2011 with reliance on a written “protocol” to teach the math snacks
  - The “protocol” gave specific instructions to teachers, such as “ask ___ when done with the animation” and “do ___ activity.” The control group did not receive the protocol
  - The same pre- and post-test was administered
    - Pre-test was administered in October, Post-test in January
  - The test included open-ended questions as well as multiple choice: What point on the number line below best represents the location of 6.4?



- The preliminary results were that nothing was different between the two groups
- Both groups had gains of 1-3 points (on a 27 point test)

Examining students' approaches

- I videoed and interviewed 3 students from a "proofs" class.
- 45 minutes were focused on the uninterrupted, think-aloud production of the proof, followed by 15 minutes of follow-up interview.
- One page of notes was given to the students starting with the definition of semigroup and supplying all information needed to prove the theorem.

The Abridged Notes and the Theorem

- A semigroup $(S,\cdot)$ is a nonempty set S together with a binary operation $\cdot$ on S such that the operation is associative. That is, for all a, b, and $c \in S$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- A nonempty subset A of a semigroup S is called an ideal of S if $AS \cup SA \subseteq A$ where $SA = \{sa | s \in S$ and $a \in A\}$.
- An ideal K of a semigroup S which does not properly contain any other ideal of S is called a minimal [left, right] ideal of S.
- Theorem: Every semigroup has at most one minimal ideal.

Results

- No one finished the proof correctly after 45 min. One student finished, but with some gaps in her proof.
- Every student immediately considered a semigroup S, and all approached the proof by assuming two, or n, minimal ideals.
- After this, each student proved the theorem differently, but that did not mean more logic was used.

Motivation and Questions

- These 3 students should have been able to prove the theorem but could not in the 45 minute interview.
- All 3 “got stuck” during the interview.
- How can people be observed constructing proofs alone (with unlimited time)?
- Do mathematicians “get stuck” and how do they get “un-stuck?”

Background Literature

- Mathematicians' knowledge
  - Actions during proof validations (Weber, 2008)
  - Mathematicians' learning (Burton, 1999; Wilkerson-Jerde & Wilensky, 2011)
  - Using diagrams to construct proofs (Samkoff, Lai, & Weber, 2011)
- Students' proving
  - Difficulties (Moore, 1994; Weber & Alcock, 2004)
  - Validations of proofs (Selden & Selden, 2003)
  - Comprehension of proofs (Conradie & Frith, 2000; Mejia-Ramos, et al., 2010)
  - Proof schemes (Harel & Sowder, 1998)

Impasses

- Impasse – A period of time when a prover feels or recognizes the argument is not progressing and he or she has no new ideas
  - Also known as “getting stuck” or “spinning one’s wheels”
  - Different from an impasse defined for automated computer provers (Meier & Melis, 2005)
- Two kinds of actions to recover from an impasse
  - Mathematical or non-mathematical

Incubation

- Incubation – a period of time, following a proof attempt, during which similar activity does not occur
- The second stage of the 4 stages of creativity (Wallas, 1926)
    - Preparation, Incubation, Illumination, Verification
- Poincare, Hadamard, and other mathematicians have described a period of incubation, followed by an “insight”
- Apparently problem solvers should have interest in finding the solution for incubation to have any effect

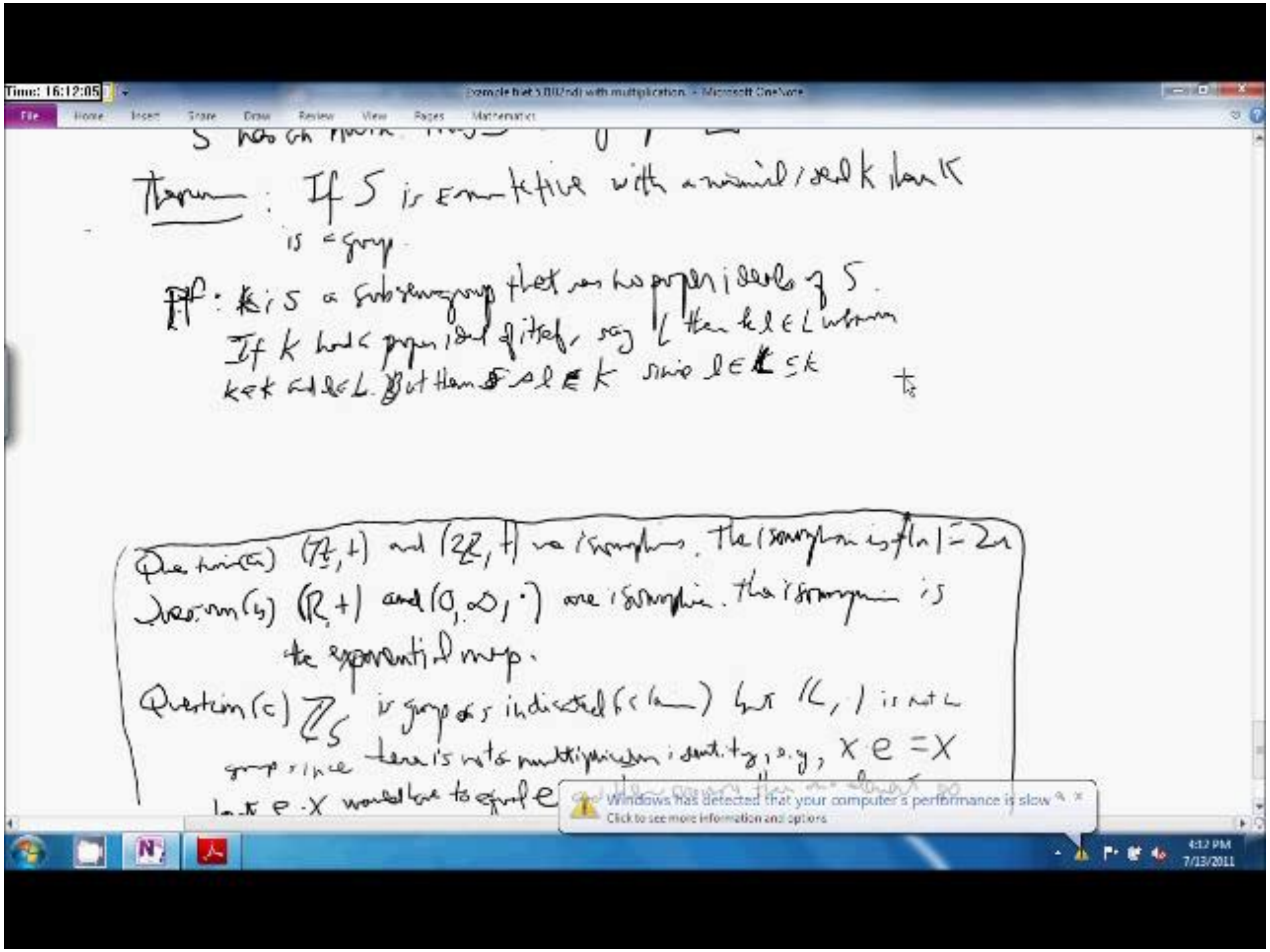
Participants and Tasks

- Nine research mathematicians (3 algebraists, 2 analysts, 3 topologists, 1 logician)
  - Eight males, one female
- Tasks – prove theorems in notes on semigroups (10 definitions, 13 theorems, 7 example requests, and 4 questions)
- Chosen for two reasons
  - Material (I hoped) was unfamiliar but accessible
  - Last two theorems require non-obvious lemmas and were difficult for students

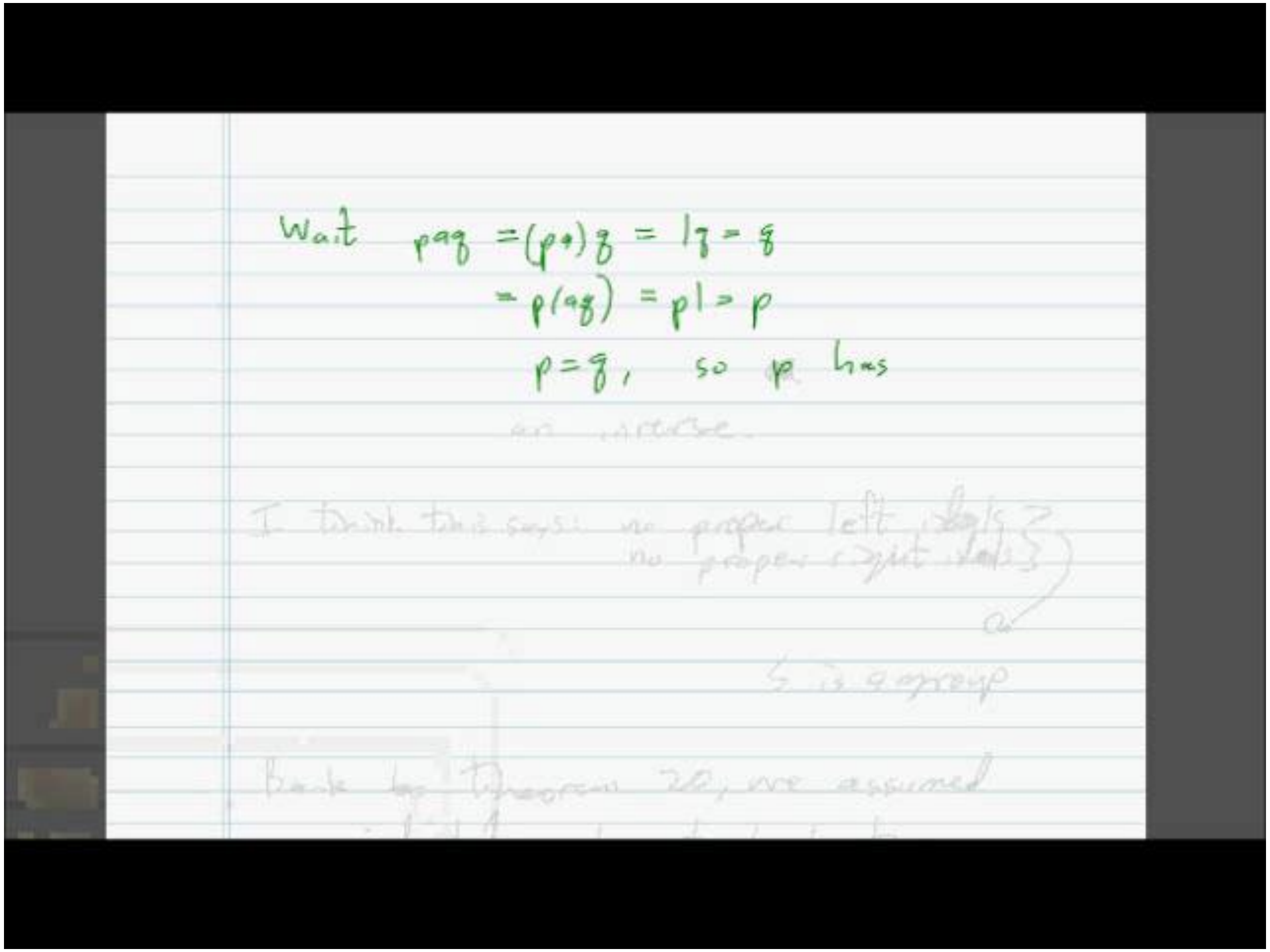
Data Collection

- Electronically:
  - The first four mathematicians proved on a tablet PC, set-up with CamStudio (screen-capturing software) and OneNote (space for their writing).
  - The final five mathematicians proved with a LiveScribe pen and special paper, capable of recording audio and writing in real-time.
- Both had time and date stamps for each writing session
- Advantages:
  - Used at the participant’s leisure
  - Real-time recording of the proving process
  - Never done before

Example of Tablet PC



Example of LiveScribe pen



Data Collection, cont.

- Each mathematician kept the equipment for 2-7 days.
- I analyzed the screen captures and the proof attempts.
- One or two days later, I interviewed the mathematicians about their proofs and their proving attempts.
- I also had two videoed “focus group” sessions: one for the tablet participants, the other for the LiveScribe pen participants.
- Two mathematicians volunteered the choice of semigroups was judicious:
  - Grasp concepts quickly
  - At least one of the theorems was challenging to prove

Summary Data

- 4 of the 9 professors had problems with the equipment, and thus did not produce “live” data
- 6 of the 9 professors had impasses with at least one of the last two theorems
- Average time of a professor’s work on the technology: 2 hours, 5 minutes
- Average time from first technology time stamp until the last: 19 hours, 56 minutes
- Average amount of pages written: Around 13

MICHIGAN STATE UNIVERSITY

Dr. A

- Applied analyst
- Encountered impasse with the final theorem in the notes: “If S is a commutative semigroup with minimal ideal K, then K is a group.”
- Done on a tablet PC
- Total time: 22 hours, 17 minutes (July 13, 2:44 PM - July 14, 1:01 PM)

Proving Process of Dr. A, Day 1

- 3:48 PM Attempted a proof of Theorem 21 by contradiction
- 3:54 PM Moved on to the final part of the notes containing a request for examples
- 4:05 PM Scrolled on the screen back up to view his first proof attempt, which he then erased.
- 4:12 PM Attempted the proof again

Proving Process of Dr. A, Day 2

- Next screen capture at 11:07 AM of Day 2.
  - Used mappings and inverse mappings of elements
- “I don’t know how to prove that K itself is a group.”
- After a 33-minute gap, he proved the theorem successfully.

Dr. A's Exit Interview

- Dr. A acknowledged his impasse:
  - "One has to show there aren't any sub-ideals of the minimal ideal itself, considered as a semigroup, and that's where I got a little bit stuck."
- Dr. A gets out of this impasse (consciously) by walking around and doing his departmental duties.

Actions to Overcome Impasses

- Viewing the impasses, the action to overcome is either mathematical or non-mathematical
- All the actions to overcome impasses are accompanied by exit interview quotes from professors supporting the action

Mathematical Actions

- Using methods that occurred earlier in the session
  - "It would be fairly easy to prove…it's likely an argument, kind of like the one I already used…" (Dr. H)
- Using prior knowledge from their own research
  - "I'm trying to think if there's anything in the work that I do that...I mean some of the stuff I've done about subspaces of $L^2(\mathbb{R})$, umm...there are things called principal shift invariance spaces that the word principal comes into play." (Dr. A)

Mathematical Actions, cont.

- Using a database of proving techniques
  - "Your brain is randomly running through arguments you've seen in the past… standard techniques that keep running through my head, sort of like downloading a whole bunch at the same time and figuring out which way to go." (Dr. F)
- Doing other problems and coming back to their impasse
  - "I moved on because I was stuck...maybe I was going to use one of those examples...I might get more information by going ahead." (Dr. B)
- Doing mathematics unrelated to the impasse subject
  - "What I try to do is to keep three projects going…I make them in different areas and different difficulty levels…" (Dr. E)

Non-mathematical Actions

- Walking
  - “When I’m stuck, I often feel like taking a break. And indeed, you come back later and certainly for a mathematician you go off on a walk and you think about it.” (Dr. G)
- Doing tasks unrelated to mathematics
  - “Yeah I’ll do something else, and I’ll just do it, and if there’s a spot where I get stuck or something, I’ll put it down and I’ll watch TV, I’ll watch the football game, or whatever it is, and then at the commercial I’ll think about it and say yeah that’ll work…” (Dr. E)

Non-mathematical Actions, cont.

- Going to lunch/eating
  - "So I had spent probably the last 30 min to an hour on that time period working on number 20 going in the wrong direction. Ok, so I went to lunch, came back, and while I was at lunch, I wasn't writing or doing things, but I was just standing in line somewhere and it occurred to me the…(laughs)…how to solve the problem." (Dr. B)
- Waking up
  - "It often comes to me in the shower…you know you wake up, and your brain starts working and somehow it just comes to me. I've definitely gotten a lot of ideas just waking up and saying "That's how I'm going to do this problem." (Dr. F)

The importance of non-mathematical actions

- Dr. G, from the focus group session: “When we are working on something, we are usually scribbling down on paper. When you go take a break,… you are thinking about it in your head without any visual aides….[walking around] forces me to think about it from a different point of view, and try different ways of thinking about it, often global, structural points of view.”
- Dr. F: “You just come back with a fresh mind. You’re zoomed in too much and you can’t see anything around it anymore.”
- Dr. A: “I do have a belief that if I walk away from something and come back it’s more likely that I’ll have an idea than if I just sit there.”

Discussion

- Educators want their students to have that “Eureka” or “AHA!” moment (Liljedahl, 2004)
- Incubation is important to mathematicians, so how can we show this effect to our students?
- One way might be to introduce “good” problems that require a good amount of thought.
- Schoenfeld (1982) described a “good” problem:
  - The problem needs to be accessible. That is, it is easily understood, and does not require specific knowledge to get into.
  - The problem can be approached from a number of different ways.
  - The problem should serve as an introduction to important mathematical ideas.
  - The problem should serve as a starting point for rich mathematical exploration and lead to more good problems.

Future research (cont.)

- Compare graduate students' data to that of the mathematicians in the proving process
- Possibly use Carlson and Bloom's (2005) taxonomy of the cyclic nature of problem solving with proof-writing
- Using the data collection technique to help students' proving approaches
  - Akin to sports' "film sessions"
  - May be helpful in transition-to-proof or other proof-based courses

Future research (cont.)

- I would like to transfer the Math Snacks idea to secondary math education, using animations and games that just help introduce key concepts in pre-algebra or the idea of a variable
- I would also examine how games are being implemented in the schools, and whether and what kind of lessons built around games are effective
- Finally, I would like to create eTextbooks that utilize animations, games, and interactive tools in undergraduate mathematics (e.g., calculus, topology, algebra, etc.)

References

- Ayalon, M., & Even, R. (2008). Deductive reasoning: In the eye of the beholder. *Educational Studies in Mathematics 69*, 235-247.
- Ayalon, M., & Even, R. (2008). Views of mathematical educators on the role of mathematics learning in the development of deductive reasoning. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rotano, & A. Sepulveda, *Proceedings of the joint meeting of PME 32 and PME-NA XXX, Vol. 2* (pp. 113-120). Mexico: Cinvestav-UMSHN.
- Baker, S. (2001). *Proofs and logic: An examination of mathematics bridge course proofs.* Cookeville, TN: Tennessee Technological University.
- Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37*, 121-143.
- Carlson, M., & Bloom, I. (2005). The cyclic nature of problem solving. *Educational Studies in Mathematics 58*, 45-75.
- Chin, E.-T., & Tall, D. (2003). Mathematical proof as formal procept in advanced mathematical thinking. In *International group for the psychology of mathematics education, paper presented at the 27th international group for the psychology of mathematics education conference held jointly with the 25th PME-NA conference* (pp. 213-220). Kirstenhof, Cape Town, 7945, South Africa: International Group for the Psychology of Mathematics Education.

References (cont.)

- Conradie, J., & Frith, J. (2000). Comprehension tests in mathematics. *Educational Studies in Mathematics, 42*, 225-235
- Downs, M. L., & Mamona-Downs, J. (2005). The proof language as a regulator of rigor in proof, and its effect on student behavior. In M. Bosch, *Proceedings of CERME 4, working group 14: Advanced mathematical thinking* (pp. 1748-1757). Sant Feliu de Gix, Spain: FUNDEMI JQS - Universitat Ramon Llull.
- Epp, S. S. (2003). The role of logic in teaching proof. *MAA Monthly*, 886-899.
- Hadamard, J. (1945) *The Mathematician's Mind*. Princeton University Press.
- Hanna, G., & de Villiers, M. (2008). ICMI study 19: Proof and proving in mathematics education (Discussion document). *ZDM-The international journal of mathematics education, 40(2)*, 329-336.
- Krashen, S. (2001). Incubation: A neglected aspect of the composing process? *ESL Journal, 4*, 10-11.
- Meier, A., & Melis, E. (2006). Impasse-driven reasoning in proof planning. In M. Kohlhase (Ed.), *Mathematical Knowledge Management: 4th International Conference MKM 2005* (pp. 143-158). Berlin: Springer-Verlag.

References (cont.)

- Mejia-Ramos, J. P., Weber, K., Fuller, E., Samkoff, A., Search, R., & Rhoads, K. (2010). Modeling the comprehension of proof in undergraduate mathematics. *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1-22). Raleigh, NC.: Available online.
- Moore, R. (1994). Making the transition to formal proof. *Educational Studies in Mathematics 27*, 249-266.
- Rips, L. J. (1994). *The psychology of proof: Deductive reasoning in human thinking.* Cambridge, MA: The MIT Press.
- Samkoff, A., Lai, Y., & Weber, K. (2011). How mathematicians use diagrams to construct proofs. *Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 430-444). Portland, OR.: Available Online.
- Selden, A., & Selden, J. (1999). *The role of logic in the validation of mathematical proofs.* Cookeville, TN: Tennessee Technological University.
- Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education, 34*, 4-36.

References (cont.)

- Selden, J., & Selden, A. (2009). Teaching proof by coordinating aspects of proofs with students' abilities. . In D. A. Stylianou, M. L. Blanton, & E. J. Knuth, *Teaching and learning proof across the grades* (pp. 339-354). New York, NY: Taylor & Francis.
- Sio, U. N., & Ormerod, T. C. (2009). Does incubation enhance problem solving? A meta-analytic review. *Psychological Bulletin, 35*, 94-120.
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.
- Wallas, G. (1926) *The Art of Thought*. New York: Harcourt, Brace and Company.
- Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education, 30*, 431-459.
- Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*, 209-234.
- Wilkerson-Jerde, M. H., & Wilensky, U. J. (2011). How do mathematicians learn math? Resources and acts for constructing and understanding mathematics. *Educational Studies in Mathematics, 78*, 21-43.

Acknowledgments, questions and comments

- Acknowledgments: Thanks to all the professors and graduate students who served as participants in this research. Thanks to everyone in the Math Snacks team for their work. Finally, thanks to Dr. and Dr. Selden for their advising.
- If you have further questions, please contact me at savic@msu.edu or visit www.milossavic.com. Thank you!