AN EXAMINATION OF PROVING USING A PROBLEM-SOLVING FRAMEWORK

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A link between proving and problem solving has been well established in the literature (Furinghetti & Morselli, 2009; Weber, 2005). *In this paper, I discuss similarities and differences between proving and problem solving by using the Multidimensional Problem-Solving Framework created by Carlson and Bloom* (2005) *on Livescribe pen data from a study of proving* (Savic, 2012). *I focus on two participants' proving processes: Dr. G, a topologist, and L, a mathematics graduate student. Many similarities were revealed by using the Carlson and Bloom framework, but also some differences distinguish the proving process from the problem-solving process. In addition, there were noticeable differences between the proving of the mathematician and the graduate student. This study may influence a proving-process framework that can encompass both the problem-solving aspect of proving and the differences found.*

Key words: Proof, Proving, Proof construction, Problem solving

Proof and proving are central to advanced undergraduate and graduate mathematics courses, yet there is little discussion in these courses of the proving process behind the proofs presented. Since there is an overlap between proving and problem solving (Furinghetti & Morselli, 2009, Weber, 2005), one might look at the problem-solving literature in order to describe some of the aspects of the proving process. I used the Multidimensional Problem-Solving Framework created by Carlson and Bloom (2005) in order to examine the proving processes of a topologist, Dr. G, and a mathematics graduate student, L. I discuss the adequacies and limitations of their framework for describing the proving processes.

Background Literature

Selden, McKee, and Selden (2010) stated that the proving process "play[s] a significant role in both learning and teaching many tertiary mathematical topics, such as abstract algebra or real analysis" (p. 128). In addition, professors teaching upper-division undergraduate mathematics courses often seem to ask students to produce original proofs to assess their understanding.

In the mathematics education literature, there are several analytical tools about proof production or the proving process, including "proof schemes" (students' ways of "ascertain[ing] for themselves or persuad[ing] others of the truth of a mathematical observation") (Harel & Sowder, 1998, p. 243), affect and behavioral schemas (habits of mind that further proof production) (Furinghetti & Morselli, 2009; Douek, 1999; Selden, McKee, & Selden, 2010) and semantic or syntactic proof production (Weber & Alcock, 2004). Both aspiring and current mathematicians seem to need flexibility in their proving styles in order to be successful in mathematics (Weber, 2004). Yet, there seems to be no overall proving-process theoretical framework that encompasses most of the above ideas about the proving process.

Past research has indicated connections between proving and problem solving, usually citing proving as a subset of problem solving. In Furinghetti and Morselli (2009), the authors stated that "proof is considered as a special case of problem solving" (p. 71). In Weber's (2005) paper, he considered "proof from an alternative perspective, viewing proof construction as a problem-solving task" (p. 351). This connection influenced my research questions: Can Carlson and

Bloom's (2005) Multidimensional Problem-solving Framework be used to describe the proving process? If changes or additions are called for, what might they be?

Carlson and Bloom's Multidimensional Problem-Solving Framework

The Multidimensional Problem-Solving Framework described by Carlson and Bloom (2005) has four phases: orienting, planning, executing, and checking. Each of these phases can be associated with four attributes: resources, heuristics, affect, and monitoring.

The orienting phase includes "the predominant behaviors of sense-making, organizing and constructing" (p. 62). The planning phase is when a participant "appeared to contemplate various solution approaches by imaging the playing-out of each approach, while considering the use of various strategies and tools" (pp. 62-63). In addition, during the planning phase Carlson and Bloom often observed an additional sub-cycle consisting of (a) *conjecturing* a solution, (b) *imagining* what would happen using the conjectured solution, and (c) *evaluating* that solution. The executing phase involved "mathematicians predominantly engaged in behaviors that involved making constructions and carrying out computations" (p. 63). Finally, the checking phase was observed when the participants verified their solutions.

Carlson and Bloom (2005) stated that "the mathematicians rarely solved a problem by working through it in linear fashion. These experienced problem solvers typically cycled through the plan-execute-check cycle multiple times when attempting one problem" (p. 63). They also stated that this cycle had an explicit execution, usually in writing, and formal checking that used computations and calculations that were also in writing.

Research Setting

One topologist, Dr. G, and one graduate student, L, were given a set of notes on semigroups and a Livescribe pen and paper, capable of capturing both audio and real-time writing using a small camera the near end of the ballpoint pen. These were two participants of a larger study of nine mathematicians and five graduate students (Savic, 2012), who were asked to answer two questions, provide seven examples, and prove thirteen theorems in the notes. Dr. G used the Livescribe equipment for proving or answering for a total of five hours and 31 minutes, while L used the equipment for a total of three days, 22 hours, and 11 minutes. I selected Dr. G's data because he spoke a significant amount of the time while proving and also encountered impasses when proving this theorem. I chose L's data because he was one of only two graduate students who attempted a proof of this theorem and I hoped that his transcript would be amenable to analysis using the Carlson and Bloom framework. I focused the coding of the proving processes on the theorem, "*Theorem 20: A commutative semigroup with no proper ideals is a group.*" The audio/video recordings were transcribed so that the audio and actions on the paper corresponded. Once the sessions were transcribed, coding was done with the Carlson and Bloom (2005) framework. A sample of the transcription, with coding, can be seen in Table 1.

Table 1: Coding the proving process of Dr. G

Time	Writing/Speaking	Coding
7:02 AM	Th 20: A comm semigp w/ no proper ideals is a gp.	Orienting (Resources)
7:03 AM	Hmm . . . I'm taking a break, breakfast, etc. Back to this later. Must think on this.	Orienting (Resources)
	BREAK 7:04 AM - 8:07 AM	
8:07 AM	Ok, I thought about this while on a cold walk in the fog.	Planning (Heuristics)

	Pf: Given $g \in S$, a semigp., consider the ideal g. . . . (Then he stops and puts “comm.” between “a” and “semigp.”) . . . $gS = \{gs \mid s \in S\}$. Since S has no proper ideals, $gS = S$, so $\exists g^{-1} \in S \ni gg^{-1} =$ (32 second pause, then he strikes through the whole proof)	Executing (Resources) Checking (Monitoring) Executing (Resources) Checking (Monitoring)
8:09 AM	First need an identity, not given. (Then he goes back to the expression “$gg^{-1} =$” and writes a question mark with a circle around it.) Turn page.	Checking (Resources), Planning (Resources) Cycling back (Back to planning, re-approaching to answer why there is no identity)

A Description of the Coding of the Sample

At 7:02 AM Dr. G started the proving period of Theorem 20; he had proved the theorems in rest of the notes in the two hours prior to this first attempt. This was his first proving period on Theorem 20. He wrote the theorem on paper, probably orienting himself to what he needed to prove. There was a one-minute pause, which I infer that he was planning or orienting himself to the theorem. Since he had proved Theorems 13-19 quite quickly, his decision to take a walk at 7:04 AM might have been because he had not quickly seen how to attempt this proof. I assume that during the break he might have been planning how to prove Theorem 20, probably using the conjecture-imagine-evaluate cycle mentioned above. At 8:07 AM, he started executing the idea that he had generated during the walk. He corrected his work by inserting “comm.” to be precise, something that I coded as “checking” and “monitoring” for correctness. Then Dr. G went back to executing his idea, using an element, g, in the semigroup and multiplying it by the whole semigroup to create an ideal. There was a 32-second pause, and then he crossed out the entire proof that he had just written. This was coded as checking. In fact, at 8:09 AM, he wrote why he crossed out this proof attempt: he needed an identity, which had not been given. I coded this as “checking (resources),” because Dr. G apparently used what he knew about groups to verify this attempt. He then cycled back to planning, because there was a 95-minute gap before he wrote something else, beginning with a different idea, and eventually re-orienting himself.

Similarities and Differences Found when Using the Problem-solving Framework

For most parts of the transcripts, the Multidimensional Problem-Solving Framework could be used to adequately describe the proving process. For example, there were multiple situations in both the transcripts of Dr. G and L that involved both the conjecture sub-cycle (conjecturing, imagining, evaluating) and the larger cycle of planning, executing, and checking. For example, in Table 2, L can be seen using the planning-executing-checking cycle.

Table 2: Coding the proving process of L

10:19 AM	First we want to show S has an identity 1.	Planning (Heuristics)

	(pauses for 45 sec)	Planning (Cycling)
10:20 AM	(pauses for 20 sec) If possible. Suppose S has no identity. Then for every $a \in S, ab \neq a$ for all	Planning (Monitoring) Executing (Heuristics) Executing (Resources)
10:21 AM	(pauses for 25 sec, then crosses out "Then for every $a \in S, ab \neq a$ for all") Let $a \in S$. Let $A = \{ab : b \in S, ab \neq a\}$.	Checking (Monitoring), Planning (Resources) Executing (Resources)

However, one can also see in L's, and especially in Dr. G's, transcripts five aspects of proving that are not covered in the problem-solving framework and perhaps should be included in a proving framework: (1) *Re-orienting* is not a part of the problem-solving framework, but appears to sometimes occur in proving. At 9:44 AM Dr. G wrote he was "suspicious that [the theorem] was [not] true." He thought about finding examples to show this, which seems different from the original orientation of preparing to write a proof. (2) Dr. G also experienced an *impasse*, a time during which he saw he was not progressing and had no more new ideas. This led to a period of *incubation* during which he did not pursue the proof and later to an *insight* leading to the proof. The larger study of the nine mathematicians also suggests impasses and incubation are important in proving. (3) *Planning and checking can be an integral part of executing*. For example, in Table 2, L hesitates for 25 seconds, crosses out his previous attempt (which can be taken as a sign of checking), and immediately proceeds with another approach. The writing of the new idea can be seen as executing, but any planning, such as formulating the new idea, must have occurred simultaneously with the earlier checking or with the later executing. Also, when L finished the proof of the theorem, he immediately went on to prove the next theorem, thus apparently not completing the full planning-executing-checking cycle. (4) Planning for Dr. G and L seems to be of two kinds, *global* and *local*. By *global planning* I mean planning the overall structure of a proof, for example, L's deciding to prove a theorem by contradiction. By *local planning* I mean planning how to proceed on a small part of a proof. In some cases, this can consist of "exploring," that is, collecting new information without knowing how, or whether, it will be useful. In Theorem 20, one can find that $ax = b$ is solvable for x without immediately knowing how, or whether, this might be useful. (5) Checking can also be *local*, for example, checking whether a definition has been properly applied, or *global*, that is, checking whether one's entire proof is sound.

Discussion and Limitations

Carlson and Bloom's framework describes the process of problem solving and some aspects of the proving process well. When posed a problem like those in the Carlson and Bloom study, mathematicians can rather easily and quickly get conversant with the constraints (orienting) and then go about solving the problem (planning-executing-checking). In my study, however, the responsibility was on the participants to figure out newly introduced concepts while proving theorems that they did not always consider to be true. Carlson and Bloom were also in the room when the interview took place and were able to take notes on their participants' behavior, so observations of affect and heuristics were easily witnessed. My participants, on the other hand, were allowed to take the notes home and work on the proofs whenever they pleased, which gave them a naturalistic setting but did not allow me to view these attributes personally. Finally, Dr. G took long breaks, which may have included incubation (Savic, 2012), in his proving process. Incubation was not observed in the Carlson and Bloom (2005) paper perhaps due to the time constraints of an in-person interview, but was extremely helpful in the proving process.

Using Carlson and Bloom's (2005) framework, I was also able to distinguish some differences between the proving processes of Dr. G and L. For example, Dr. G asked more questions (both in writing and verbally) about the structure of and the constraints of the theorem. L never went back to orienting himself after the first few minutes. One advantage to using notes, like those on semigroups, is that they are easily accessible to novice, intermediate, and advanced provers, while providing challenging theorems to prove. Could a proving-process framework be created to include the Carlson and Bloom framework and also the differences noted? Also, is it beneficial that professors have such a framework to diagnose problems with a student's proving process?

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