AN EXAMINATION OF PROVING USING A PROBLEM-SOLVING FRAMEWORK

Milos Savic
Michigan State University

A link between proving and problem solving has been well established in the literature (Furinghetti & Morselli, 2009; Weber, 2005). In this paper, I discuss similarities and differences between proving and problem solving by using the Multidimensional Problem-Solving Framework created by Carlson and Bloom (2005) on Livescribe pen data from a previous study of proving (Savic, 2012). I focus on two participants' proving processes: Dr. G, a topologist, and L, a mathematics graduate student. Many similarities were revealed by using the Carlson and Bloom (2005) framework, but also some differences distinguish the proving process from the problem-solving process. In addition, there were noticeable differences between the proving of the mathematician and that of the graduate student. This study may influence a proving-process framework that can encompass both the problem-solving aspects of proving and the differences found.

Key words: Proof, Proving, Proof construction, Problem solving

Proof and proving are central to advanced undergraduate and graduate mathematics courses, yet there is little discussion in these courses of the proving process behind the proofs presented. Since there is an overlap between proving and problem solving (Furinghetti & Morselli, 2009, Weber, 2005), one might look at the problem-solving literature in order to describe some of the aspects of the proving process. I used the Multidimensional Problem-Solving Framework created by Carlson and Bloom (2005), coupled with a data collection technique (Savic, 2012) specifically aimed at collecting the real-time actions that a prover takes, in order to examine the proving processes of a topologist, Dr. G, and a mathematics graduate student, L. I discuss the adequacies and limitations of their framework for describing the observed proving processes. I also discuss the noticeable differences between Dr. G's and L's proving actions. Finally, I conjecture some educational strategies that might be useful for making some implicit actions in the proving process explicit.

Background Literature

Selden, McKee, and Selden (2010) stated that the proving process "play[s] a significant role in both learning and teaching many tertiary mathematical topics, such as abstract algebra or real analysis" (p. 128). In addition, professors teaching upper-division undergraduate mathematics courses often seem to ask students to produce original proofs to assess their understanding. When producing an original proof, some naïve students might not know where to start or how to handle the proving process. This study may help in designing a proving-process framework, which can then be used a tool by students in their own struggles with the proving process.

Both aspiring and current mathematicians seem to need flexibility in their proving styles in order to be successful in mathematics (Weber, 2004; Iannone, 2009). In the mathematics education literature, there are several analytical tools concerning proof production or the proving process, including "proof schemes" (students' ways of "ascertain[ing] for themselves or persuad[ing] others of the truth of a mathematical observation") (Harel & Sowder, 1998, p. 243), affect and behavioral schemas (i.e., habits of mind that further proof production) (Furinghetti & Morselli, 2009; Douek, 1999; Selden, McKee, & Selden, 2010) and semantic or syntactic proof production (Weber & Alcock, 2004).

One analytical tool for the proving process focuses more on the problem-solving aspect while other aspects, according to the authors, may be "autonomous" in proving. Selden and

Selden (2009) described two aspects of a written proof, the formal-rhetorical part and the problem-centered part. According to the authors:

> The *formal-rhetorical* part of a proof (what we have also referred as the *proof framework*) is the part of a proof that depends only on unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results . . . . The remaining part of a proof [is] the *problem-centered* part . . . that *does* depend on genuine problem solving, intuition, and a deeper understanding of the concepts involved. (Selden & Selden, 2013, p. 6)

This problem-centered part can be considered as the part of the proof that uses problem-solving, in the sense of Schoenfeld (1985), who stated that a *problem* is a mathematical task for an individual if that person does not already know a method of solution for that task. Past research has indicated connections between proving and problem solving, usually citing proving as a subset of problem solving. In Furinghetti and Morselli (2009), the authors stated that "proof is considered as a special case of problem solving" (p. 71). Weber (2005) considered "proof from an alternative perspective, viewing proof construction as a problem-solving task" (p. 351).

Polya (1957) described many ways to go about problem solving that have been summarized into four overarching steps: "(i) understanding the problem, (ii) developing a plan, (iii) carrying out the plan, and (iv) looking back." Schoenfeld (1992), somewhat influenced by Polya, described six processes when doing a problem-solving activity: "read, analyze, explore, plan, implement, and verify" (p. 61). Carlson and Bloom (2005) utilized both Polya's and Schoenfeld's ideas in creating their multidimensional problem-solving framework, which I describe in the next section.

Carlson and Bloom's Multidimensional Problem-Solving Framework

The Multidimensional Problem-Solving Framework described by Carlson and Bloom (2005) has four phases, each with the same four associated problem-solving attributes. The four phases are orienting, planning, executing, and checking. The four associated problem solving attributes are resources, heuristics, affect, and monitoring. The table below (Figure 1), taken from Carlson and Bloom (2005, p. 67), shows the multidimensional aspects of their framework.

Phase • Behavior	**Resources**	**Heuristics**	**Affect**	**Monitoring**
Orienting • Sense making • Organizing • Constructing	Mathematical concepts, facts and algorithms were accessed when attempting to make sense of the problem. The solver also scanned her/his knowledge base to categorize the problem.	The solver often drew pictures, labeled unknowns and classified the problem. (Solvers were sometimes observed saying, "this is an X kind of problem.")	Motivation to make sense of the problem was influenced by their strong curiosity and high interest. High confidence was consistently exhibited, as was strong mathematical integrity.	Self-talk and reflective behaviors helped to keep their minds engaged. The solvers were observed asking, "What does this mean?"; "How should I represent this?"; "What does that look like?"
Planning Conjecturing Imagining Evaluating	Conceptual knowledge and facts were accessed to construct conjectures and make informed decisions about strategies and approaches.	Specific computational heuristics and geometric relationships were accessed and considered when determining a solution approach.	Beliefs about the methods of mathematics and one's abilities influenced the conjectures and decisions. Signs of intimacy, anxiety, and frustration were also displayed.	Solvers reflected on the effectiveness of their strategies and plans. They frequently asked themselves questions such as, "Will this take me where I want to go?", "How efficient will Approach X be?"
Executing • Computing • Constructing	Conceptual knowledge, facts and algorithms were accessed when executing, computing and constructing. Without conceptual knowledge, monitoring of constructions was misguided.	Fluency with a wide repertoire of heuristics, algorithms, and computational approaches were needed for the efficient execution of a solution.	Intimacy with the problem, integrity in constructions, frustration, joy, defense mechanisms and concern for aesthetic solutions emerged in the context of constructing and computing.	Conceptual understandings and numerical intuitions were employed to reflect on the sensibility of the solution progress and products when constructing solution statements.
Checking • Verifying • Decision making	Resources, including well-connected conceptual knowledge informed the solver as to the reasonableness or correctness of the solution attained.	Computational and algorithmic shortcuts were used to verify the correctness of the answers and to ascertain the reasonableness of the computations.	As with the other phases, many affective behaviors were displayed. It is at this phase that frustration sometimes overwhelmed the solver.	Reflections on the efficiency, correctness and aesthetic quality of the solution provided useful feedback to the solver

Figure 1: Carlson and Bloom's Multidimensional Problem-Solving Framework (2005, p. 67)

Below I describe each phase, as well as the problem solving attributes associated with each phase. All phases and attributes by Carlson and Bloom (2005) emerged during their analysis of the problem-solving processes of eight research mathematicians and four Ph.D. candidates. An example of one of the problems posed in their study was

Problem 1: A square piece of paper $ABCD$ is white on the front side and black on the back side and has an area of 3 in.^{2} Corner A is folded over to point A' which lies on the diagonal AC such that the total visible area is ½ white and ½ black. How far is A' from the fold line? (Carlson & Bloom, 2005, p. 71)

Orienting

According to Carlson and Bloom (2005), the orienting phase includes "the predominant behaviors of sense-making, organizing and constructing" (p. 62). Examples of this phase in their study included defining unknowns, sketching a graph, or constructing a table. They stated that an individual may execute these orienting actions with "intense cognitive engagement," ultimately understanding the nature of the problem. Use of resources in the orienting phase can include accessing mathematical concepts, facts, and algorithms. Use of heuristics in the orienting phase can include drawing pictures, labeling unknowns, and classifying the problem. Affect experienced during the orienting phase can include motivation to make sense of the problem, high confidence, and strong mathematical integrity. Finally, use of monitoring in the orienting phase can include self-talk and other reflective behaviors during sense-making, such as asking "What does this mean?"

Planning

Carlson and Bloom (2005) coded a planning phase in a transcript when a participant "appeared to contemplate various solution approaches by imaging the playing-out of each approach, while considering the use of various strategies and tools" (pp. 62-63). In addition, they often observed a subcycle of (a) *conjecture* of a solution, (b) *imagining* what would happen using the conjectured solution, and (c) *evaluating* the validity of that solution during planning phases (See Figure 2).

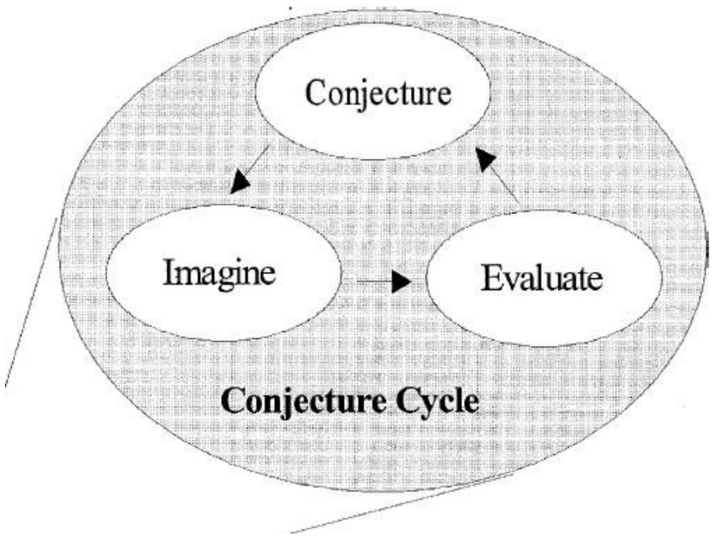

Figure 2: The conjecture-imagine-evaluate [sub]cycle
(Carlson & Bloom, 2005, p. 54)

In Carlson and Bloom's (2005) analysis, this subcycle could be exhibited by their participants either verbally or silently, but the entire planning phase occurred before the executing phase commenced. Resources used during the planning phase included conceptual knowledge and other facts needed to construct conjectures. Heuristics used, if visible to the researchers, included computations and geometric relationships. Affect exhibited by participants during the planning phase included beliefs about the methods or conjectures being employed and about their own abilities to solve the current problem. Monitoring exhibited by Carlson and Bloom's (2005) participants during the planning phase included self-reflection about the effectiveness of their current strategies.

Executing

Carlson and Bloom (2005) noted that the executing phase involved "mathematicians predominantly engaged in behaviors that involved making constructions and carrying out computations" (p. 63). Specific examples included "writing logically connected mathematical statements," using concepts and facts, and using procedures or other computations. Resources used were the same concepts, facts, and procedures that had been used during the prior planning phase. Heuristics used during the execution of the solution included fluency with the algorithms and approaches employed. Affect exhibited in the executing process involved some emotional responses to the attempted solution, such as "intimacy with the problem, frustration, joy, defense mechanisms, and aesthetics in the solution" (p. 67). Monitoring involved the participants having some sensitivity to the progress of their solutions.

Checking

The checking phase was observed when the participants verified their solutions. These behaviors included "spoken reflections by the participants about the reasonableness of the solution and written computations. . .contemplating whether to accept the result and move to the next phase of the solution, or reject the result and cycle back" (Carlson & Bloom, 2005, p. 63). Resources used during the checking phase involved "well-connected conceptual knowledge" for the "reasonableness" of their solutions. Heuristics used included knowledge of "conceptual and algorithmic shortcuts." Affect during the checking phase was similar to other affective behaviors, but frustration might overtake a participant if the solution was incorrect. Monitoring during this phase involved thinking about the "efficiency, correctness, and aesthetic quality of the solution" (Carlson & Bloom, 2005, p. 63).

The cycle of problem solving

Carlson and Bloom (2005) stated that "it is important to note that the mathematicians rarely solved a problem by working through it in linear fashion. These experienced problem solvers typically cycled through the plan-execute-check cycle multiple times when attempting one problem" (p. 63). Carlson and Bloom (2005) also stated that the cycle had an explicit execution, usually in writing, and formal checking that used computations and calculations that were also in writing. All cues exhibited by the participants and observed by the researchers, whether written, verbal, or non-verbal, were used to distinguish between phases.

Research Questions

This connection between problem solving and proof influenced my research questions: Can Carlson and Bloom's (2005) Multidimensional Problem-Solving Framework be used to describe the proving process? If changes or additions are called for, what might they be?

Research Setting

One topologist, Dr. G, and one graduate student, L, were given a set of notes on semigroups and a Livescribe pen and paper, capable of capturing both audio and real-time writing using a small camera the near end of the ballpoint pen. These were two participants of a larger study of nine mathematicians and five graduate students (Savic, 2012), who were asked to answer two questions, provide seven examples, and prove thirteen theorems in the notes. From the first use of the Livescribe equipment for proving or answering all tasks in the notes until the last minute of equipment use, Dr. G totaled five hours and 31 minutes, while L totaled three days, 22 hours, and 11 minutes. I focused the coding of the proving processes on the theorem, "*Theorem 20: A commutative semigroup with no proper ideals is a group.*" From the first use of the Livescribe pen for their proof attempt of Theorem 20 until the last, Dr. G spent three hours and 17 minutes, while L spent 41 minutes. I selected Dr. G's data because he spoke a significant amount of the time while proving and also encountered impasses when proving this theorem. I chose L's data because he was one of only two graduate students who attempted a proof of this theorem and I hoped that his transcript would be amenable to analysis using the Carlson and Bloom (2005) framework. The audio/video recordings were transcribed so that the audio and actions on the paper corresponded. Once the sessions were transcribed, coding was done with the Carlson and Bloom (2005) framework. A sample of the transcription, with coding, can be seen in Table 1.

Table 1: Coding the proving process of Dr. G

Time	Writing/Speaking	Coding
7:02 AM	Th 20: A comm semigp w/ no proper ideals is a gp.	Orienting (Resources)
7:03 AM	Hmm . . . I'm taking a break, breakfast, etc. Back to this <u>later</u>. <u>Must think on this.</u>	Orienting (Resources)
	BREAK 7:04 AM - 8:07 AM (Planning)	
8:07 AM	Ok, I thought about this while on a cold walk in the fog. Pf: Given $g \in S$, a semigp., consider the ideal g. . . . (Then he stops and puts "comm." between "a" and "semigp.") . . . $gS = \{gs \mid s \in S\}$. Since S has no proper ideals, $gS = S$, so $\exists g^{-1} \in S \ni gg^{-1} =$ (32 second pause, then he strikes through the whole proof)	Planning (Heuristics) Executing (Resources) Checking (Monitoring) Executing (Resources) Checking (Monitoring)
8:09 AM	First need an identity, not given. (Then he goes back to the expression "$gg^{-1} =$" and writes a question mark with a circle around it.) Turn page.	Checking (Resources), Planning (Resources) Cycling back (Back to planning, re-approaching to answer why there is no identity)

A Description of the Coding of the Sample

At 7:02 AM Dr. G started the proving period of Theorem 20; he had proved the theorems in rest of the notes in the two hours prior to this first attempt. This was his first proving period on Theorem 20. He wrote the theorem on paper, probably orienting himself to what he needed to prove. There was a one-minute pause, which I infer that he was planning or orienting himself to the theorem. Since he had proved Theorems 13-19 quite quickly, his decision to take a walk at 7:04 AM might have been because he had not quickly seen how to attempt this proof. I assume that during the break he might have been planning how to prove Theorem 20, probably using the conjecture-imagine-evaluate subcycle mentioned above. At 8:07 AM, he started executing the idea that he had generated during the walk. He corrected his work by inserting “comm.” to be precise, something that I coded as “checking” and “monitoring” for correctness. Then Dr. G went back to executing his idea, using an element, g, in the semigroup and multiplying it by the whole semigroup to create an ideal. There was a 32-second pause, and then he crossed out the entire proof that he had just written. This was coded as checking. In fact, at 8:09 AM, he wrote why he crossed out this proof attempt: he needed an identity, which had not been given. I coded this as “checking (resources),” because Dr. G apparently used what he knew about groups to verify this attempt. He then cycled back to planning, because there was a 95-minute gap before he wrote something else, beginning with a different idea, and eventually re-orienting himself.

Rules for coding certain situations

For reliability, I asked two other colleagues to code three small excerpts of the transcripts using the Carlson and Bloom (2005) framework. There were certain segments that merited discussion, and we came to an agreement in all of those instances. Using their assistance, I then established a set of rules to help refine the coding process. They were as follows:

1. Both participants had instances in their proving sessions that were pauses in their work. I defined a pause as a period of time in the live data proving session during which the prover does not speak or write. I had asked the participants to prove the theorems at their own leisure with unlimited time, so I was not present to ask them contemporaneously about pauses in their proving. An important aspect of coding a pause would be: If a participant made corrections immediately after a pause, then I would code the pause as “checking.” If after a pause, the participant had an idea or could continue his progress, then I would code the pause as “planning” prior to the executing phase. I also coded participants’ pauses based on what I thought a participant was accomplishing, using my own inferences about their proving process.
    a. When a participant turned off the LiveScribe pen and turned it back on, I considered that a break. All breaks were considered “planning.” This is because almost immediately after a participant turned on the pen after a break, he or she had an idea to try, which is considered in the Carlson and Bloom (2005) framework as “executing.” For example, in Dr. G’s transcript (Table 1), the break from 7:04 AM – 8:07 AM was coded as “planning.”
    b. Many pauses during the proving process were considered “planning,” because of the “executing” phase that occurred immediately afterwards.
2. Speaking was never considered “executing.” Any phase coded as “executing” occurred within the written work, and was only coded this way when it furthered (either correctly or incorrectly) the attempted proof.
3. Any “crossing out” or elimination of any part of the “executing” phase was considered “checking.” An example of this, which occurred at 8:08 AM in Dr. G’s transcript, can be found in Figure 3 below:

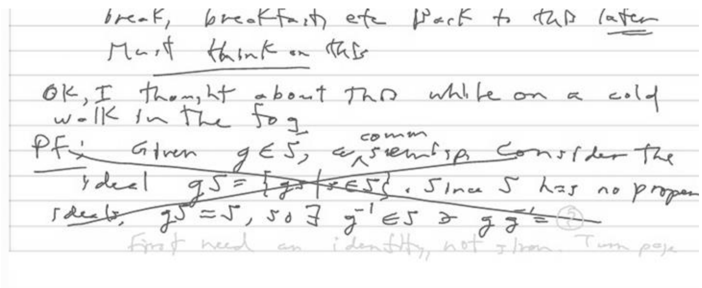

Figure 3: Dr. G’s crossed-out work on Theorem 20

Results

Carlson and Bloom's (2005) Multidimensional Problem-Solving Framework aligned well with most of what the two participants (Dr. G and L) did during the proving process. Using the phases (Orienting, Planning, Executing, and Checking) and the problem solving attributes (Resources, Heuristics, Affect, and Monitoring), I coded all of both transcripts that pertained to Theorem 20 and analyzed the situations that agreed and that disagreed with those in Carlson and Bloom's (2005) framework.

Instances of agreement with Carlson and Bloom's framework

For most portions of the transcripts, the Multidimensional Problem-Solving Framework could be used to code and describe the proving process. There were multiple situations in both transcripts that involved both the planning subcycle (conjecturing, imagining, evaluating) and the larger cycle of planning, executing, and verifying.

The planning subcycle

In Dr. G's spoken discussion of Theorem 21, "If K is a minimal ideal of a commutative semigroup S, then K is a group," he demonstrated the planning subcycle (described in the "Planning" section along with Figure 2) seen in Table 3:

Table 3: An example of the Conjecturing-Imagining-Evaluating subcycle

Time	Writing	Speaking	Coding
9:51 AM		If it [a semigroup S] has a zero element, then that [the zero element] will be a minimal ideal.	Orienting (Resources)
[Cycle Starts Here]		Does that make it a group?	Planning (Conjecturing)
		(Silence for 13 seconds)	Planning (Imagining)
		Well no,	Planning (Evaluating)
		what about the non-negative integers?	Planning (Conjecturing)

The subcycle starts with the question, "Does that make it a group?" which is a conjecture by Dr. G based on the statement, "If it [S] has a zero element, then that will be a minimal ideal." He then paused for 13 seconds. This was coded as "imagining", because Dr. G was imagining what would happen with his conjecture that the existence of a zero element, 0, forces a minimal ideal, namely $\{0\}$, to exist. The next words said by Dr. G were, "Well, no…" acknowledging that he had evaluated where he had been expecting to go with his $\{0\}$ ideal counterexample. I coded that as part of the "evaluating" phase. Finally, he ended this subcycle by conjecturing something about the non-negative integers, completing the planning subcycle. In fact, in the exit interview conducted after his proving sessions, Dr. G acknowledged that he misread the statement and was trying to prove that S was a group. Nonetheless, this episode exhibited the planning subcycle.

Example of a full cycle of planning-executing-checking

In L's proof of Theorem 20, he demonstrated the full planning-executing-checking**.}** cycle seen in Table 4. L did not speak during his proving process, so I conjectured the phases using only his written work.

Table 4: An example of the Planning-Executing-Checking cycle

10:19 AM	First we want to show S has an identity 1. (pauses for 45 sec)	Planning (Heuristics) Planning (Cycling)
10:20 AM	(pauses for 20 sec) If possible. Suppose S has no identity. Then for every $a \in S, ab \neq a$ for all	 Planning (Monitoring) Executing (Heuristics) Executing (Resources)
10:21 AM	(pauses for 25 sec, then crosses out "Then for every $a \in S, ab \neq a$ for all") Let $a \in S$. Let $A = \{ab: b \in S, ab \neq a\}$.	Checking (Monitoring), Planning Executing (Resources)

In this excerpt from L's transcript, he started the proof of Theorem 20 at 10:19 AM with "We want to show S has an identity 1." Since L was writing the sentence to tell himself of his intentions regarding the proof, this statement was coded as "planning". He then paused for a minute and five seconds, and this seemed as if he were going through the conjecturing-imagining-evaluating subcycle reflecting on how he might prove the theorem. The next statement after this pause was, "If possible." Thus, I coded this as a "planning" phase because L was changing how he wanted to approach the proof, and this required him to make sense of which proof framework he would use. After this, he wrote, "Suppose S has no identity." Since L was attempting to prove the theorem, instead of writing guiding sentences as he had done previously (at 10:19 AM), this was coded as "executing." His next sentence, "Then for every $a \in S, ab \neq a$ for all," was coded as "executing" as well. L then paused for 25 seconds. I conjectured that L was considering what he had written, so this pause was coded as "checking". At this point, he crossed out his previous work, and had another idea (namely, creating an ideal) to work with, so I coded this as "planning" right before he executed his idea, "Let $a \in S$. Let $A = \{ab: b \in S, ab \neq a\}$." Hence, the planning-executing-checking cycle can be deemed to have occurred.

Instances of difference with Carlson and Bloom's framework

Cycling back to orienting

Dr. G, after an incubation period (8:09 AM - 9:44 AM) was at a quandary about how to proceed with the proof of the theorem, and in fact had to *reorient* himself to the truth of the theorem. This can be seen in Table 5.

Table 5: An example of reorienting

Time	Writing	Speaking	Coding
8:09 AM	First need an identity, not given. (Then he goes back to the expression "$gg^{-1} =$" and writes a question mark with a circle around it.) Turn page.	None	Checking (Resources) Checking, Planning
BREAK 8:10 AM - 9:44 AM (Planning)			
9:44 AM	Later. I'm suspicious that this is true. Why should the nonexistence of proper ideals force existence of an identity?		Planning (Affect, Monitoring)

	But I don't know many examples, so I don't see a counterexample. (Silence for a minute, followed by ruffled papers, then silence)		Orienting (Resources)

After approaching the proof using a direct proof technique, Dr. G apparently thought about his approach during the incubation period. His next statement was "I'm suspicious this is not true." After this declaration, he claimed that he didn't know many examples. Generating examples for the statement of a theorem can be a way of orienting oneself to the problem of deciding on an approach for proving a theorem. But Dr. G had oriented himself once before (7:02 AM - 8:07 AM) and had already gone part way into the planning-executing-checking cycle. This is an example of an instance in proving when a prover must *reorient* himself in order to reconsider all of the information given. In fact, during his exit interview, Dr. G stated that, "as with many things, at first I thought, 'how can I prove this?' and I didn't immediately think of a proof, so I think, 'what about a counterexample?'" Dr. G was not afraid of reorienting himself with the theorem in order to create counter-examples.

Not completing a full cycle of planning-executing-checking

L finished the proof of Theorem 20 without going through the final checking phase of problem solving. This is displayed in Table 6.

Table 6: An example of not doing the final checking

Time	Writing	Coding
10:48 AM	(pauses for 25 sec, the writes next to $b^{-1} \in S$ from 10:45 AM, "For contradiction", then turns back the page, then pause for 25 sec)	Planning (Resources)
10:49 AM - 10:51 AM	Let $B = \{b \in S: b$ has no inverse$\}$. Then $B \neq \emptyset$ because $b' \in B$. So B is a proper ideal of S which is a contradiction. So every element of S has an inverse. (B is proper because $1 \notin B$). Hence S is a group.	Executing (Resources)

After he had written at 10:51 AM "Hence S is a group," he immediately proceeded to the next theorem (Theorem 21). In proving Theorem 20 (incorrectly) L had gone through the planning and executing phases, but had not performed the checking phase. There are two conjectures that I have about this. One conjecture is that he had used a technique that was very close to a technique that he had used previously when trying to prove that S has an identity, and he had taken considerable time and writing (10:39 AM – 10:45 AM) to check his work on that. The other conjecture is that he knew the end of the notes was approaching and he wanted to finish them quickly. This would be understandable, especially since he was a research Ph.D. student that had already successfully completed his comprehensive examinations, so he would probably have preferred working on his own research over proving unrelated theorems.

Discussion

Successes and limitations with the coding

The four phases of the Carlson and Bloom (2005) framework were relevant to the proving process. At first glance, the two participants (Dr. G and L) were always in one of the phases

(Orienting, Planning, Executing, and Checking) during their entire proving sessions for Theorem 20. This suggests that the four phases are very important for the proving process. This further suggests that an expansion of Carlson and Bloom's (2005) framework could potentially provide the mathematics education community a proving-process framework, complete with additional problem-solving attributes that a prover experiences. Some additional problem-solving phases may include incubation, re-orientation, and instances of multiple phases (Checking and Planning) occurring during a pause in the midst of proving or a break from proving in the proving process.

There may also be refinements of the Checking and Planning phases for a future proving-process framework. There were instances of *local planning* or proceeding on only a small part of a proof, and global planning or approaching a proof with a certain framework (Selden & Selden, 1995). An example of local planning was when Dr. G went through the planning subcycle in Table 3, where he asked himself whether a semigroup he had created was a group. An example of *global planning* was when L considered whether to prove Theorem 20 directly or by contradiction in Table 4 (10:19 AM – 10:20 AM). Checking could have also been split into *local checking* (e.g., finding minor errors) and *global checking* (i.e., seeing if a proof attempt is sound). For example, local checking occurred in the middle of Dr. G's first proof attempt when he stopped executing to write "comm." between "a" and "semigroup" (Table 1). Global checking happened about a minute later, where he crossed out his entire proof attempt (seen in both Table 1 and Figure 3).

Carlson and Bloom's (2005) framework describes the process of problem solving well. They provide ample examples from their study that support their framework. When posed a problem like those that Carlson and Bloom (2005) posed in their study (Problem 1 in the Multidimensional Problem-Solving Framework section), mathematicians can rather easily and quickly get conversant with the constraints (orienting) and then go about solving the problem (planning-executing-checking).

However, in my study, the mathematicians were given a theorem (Theorem 20), and had to go about orienting themselves. Some mathematicians (e.g., Dr. G) executed their ideas (e.g., about modifying the hypotheses) early to see where they might lead, but then had to look at the theorem again to analyze why the hypotheses needed modifying. In fact, if one assumes that a statement (in this case, a theorem) could be true or false, one must orient oneself either for a proof or for a counterexample. In mathematical problem solving, unless a problem is posed as a true or false question, some problems can often implicitly be assumed to have a solution.

Carlson and Bloom (2005) audiotaped the mathematicians in their study while they were solving the problems, and were in the room to take notes and answer questions. In my study, however, participants were given a set of notes with unlimited time and not much direction. I was unable to observe their non-verbal actions, something that I conjecture provided Carlson and Bloom (2005) considerable help with their coding. This was a limitation of my study. On the contrary, I was able to capture incubation periods and insight, which were not accounted for in the Carlson and Bloom (2005) framework. This influenced my coding. Breaks are a crucial part of creativity and problem-solving for mathematicians (Savic, 2012), yet are not considered in Carlson and Bloom's (2005) problem-solving framework.

Observed differences between the mathematician and the graduate student

When analyzing all participants in the data collection (nine mathematicians and five graduate students), I found that coding the proof attempts on Theorem 20 would give the best comparison of how one attempts a proof. Six of the nine mathematicians experienced impasses when attempting a proof of Theorem 20, but only two out of five graduate students even attempted a proof. The proof of Theorem 20 was not trivial, which provided a nice comparison between the attempts of Dr. G and L. Notice that Dr. G analyzed situations dealing with the theorem, such as "Why should the nonexistence of proper ideals force existence of an identity?" Dr. G often questioned the constraints of the hypotheses of the theorem. He went a step further and even thought that he might be able to construct a counterexample. According to my coding, L oriented himself at the beginning of the proving period for Theorem 20, and did not question the truth of the theorem, nor the constraints given. My conjecture is that the mathematician (Dr. G) has had substantial experience both with conjecturing his own theorems and adjusting the precise wording of those theorems after attempting unsuccessfully to prove them. He must have had to reorient himself rather often when engaging in mathematical research.

Another observable difference between Dr. G and L was how each handled the "checking" phase. With L, most checking phases were incorporated with the planning phase, where after multiple pauses during the proving of Theorem 20, he both crossed out a certain amount of his previous proof attempt and immediately proceeded to move to the "executing" phase. There was not an observable mixture of phases in Dr. G's proving process. This may have been due to the amount of speaking Dr. G did, which helped separate the planning and checking phases. But the more important aspect of Dr. G's checking phase was that he tried to make sense of his failed proving attempts. For example, in Table 1, Dr. G stated, "First need an identity, not given." He noticed that his previous proving attempt required an identity element, so he must adjust his next proving attempt to accommodate that requirement. During L's proving attempts, he would make minor adjustments after each attempt but had the same main idea (supposing there is no identity) in mind. Global checking may have helped L gain more information from his proving attempts.

Future Research

It would be an accomplishment if there could be a proving-process framework, similar to Carlson and Bloom's (2005) problem-solving framework. Such a framework would be helpful in assessing a student's proving and their phases or problem-solving attributes that need improvement. A teacher could isolate the phases (Orienting, Planning, Executing, and Checking) or problem-solving attributes (Resources, Heuristics, Affect, and Monitoring) that need to be worked on, and focus instruction on that phase/attribute. Also, such a framework might allow researchers to analyze their proof data to analyze and describe new phenomena that they observe.

Additionally, a data collection technique that could capture more phases and attributes would help in developing a proving-process framework. My study gathered written data in real time with synchronized audio. There was no collection of gestures, including when the participants viewed the notes to orient themselves to the statement of a theorem or to gather ideas during a planning phase. In Carlson and Bloom's (2005) study, the participants were in an interview room for a more-or-less fixed time working continuously on the problems posed. Because their participants had no time for a break or other distracting activity, their data collection technique might have influenced their participants' creativity. A combination of the two data collection techniques (LiveScribe pen and videoed interview sessions), would be much more informative, but would take more time and resources.

Finally, LiveScribe pens along with Carlson and Bloom's (2005) framework might assist students in considering the implicit actions of the proving process. For example, students could do either homework or a test with a LiveScribe pen and turn in the pen with their assignment. The professor or a graduate assistant could then upload the data and make a "movie" of a particular part of the student's proving process. Then the assignment for the students would be to code the movie using the framework. The purpose would be to expose students to certain phases and attributes, thus hoping to make the students mindful of those phases/attributes in the future. In particular, one could see that some students do not incorporate the checking phase in their problem solving or proving. An example of this phenomenon occurred with L's work in Table 6, when he finished his proof of Theorem 20 without any checking. Making phases explicit might help undergraduate students make the transition to proof-based classes quicker, thus shifting the focus of those initial proof-based courses towards their principal purpose of illuminating content.

References

Carlson, M., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent problem-solving framework. *Educational Studies in Mathematics, 58*, 45-75.

Douek, N. (1999). Some remarks about argumentation and mathematical proof and their educational implications. In I. Schwank (Ed.), *Proceedings of the First Conference of the European Society for Research in Mathematics Education, Vol. 1* (pp. 125-139). Osnabrueck: Forschungsinstitute fuer Mathematikdidaktik.

Furinghetti, F., & Morselli, F. (2009). Every unsuccessful problem solver is unsuccessful in his or her own way: Affective and cognitive factors in proving. *Educational Studies of Mathematics, 70*, 71-90.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.). *CBMS Issues in Mathematics Education, Vol. 7: Research in Collegiate Mathematics Education III* (pp. 234-283). Providence, RI: American Mathematical Society.

Iannone, P. (2009). Concept usage in proof production: Mathematicians' perspectives. In F.-L. Lin, F.-J. Hsieh, G. Hanna, & M. de Villiers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and Proving in Mathematics Education* (pp. 220-225). Taipei, Taiwan: National Taiwan Normal University.

Polya, G. (1957). *How to solve it: A new aspect of mathematical method.* Garden City, NJ: Doubleday.

Savic, M. (2012). What do mathematicians do when they reach a proving impasse? In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman, *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 531-535). Portland, OR: Online at http://sigmaa.maa.org/rume/crume2012/RUME_Home/RUME_Conference_Papers_files/RUME_XV_Conference_Papers.pdf.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Orlando, FL: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334-370). New York, NY: Macmillian.

Selden, A., & Selden, J. (2013). Proof and problem solving at university level. In M. Santos-Trigo, & L. Moreno-Armella (Eds.), *The Mathematics Enthusiast, Special Issue: International Perspectives on Problem Solving, Research in Mathematics Education, Vol. 10, 1&2* (pp. 303-334).

Selden, A., McKee, K., & Selden, J. (2010). Affect, behavioural schemas and the proving process. *International Journal of Mathematical Education in Science and Technology, 41*, 199-215.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics*, *29*, 123-151.

Selden, J., & Selden, A. (2009). Teaching proof by coordinating aspects of proofs with students' abilities. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 339-354). New York, NY: Rutledge.

Weber, K. (2004). A framework for describing the processes that undergraduates use to construct proofs. In M. J. Hoines, & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education (PME), Vol. 4* (pp. 425-432). Bergen, Norway: Bergen University College.

Weber, K. (2005). Problem-solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in the activity of proof construction. *Journal of Mathematical Behavior, 24*, 351-360.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*, 209-234.