An Examination of Proving Using a Problem-Solving Framework

RUME Conference XVI

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February 21, 2013

Motivation

- Carlson and Bloom (2005) created a Multi-Dimensional Problem-Solving Framework when explaining the problem-solving process.
- They also refer to the process of problem solving as "cyclic"
- What would happen if that framework would be used on the proving process?
  - It may assist the link between problem solving and proof.
- Also, could there be a Multi-Dimensional Proving-Process Framework?

Background Literature

- Problem solving
  - Polya, 1957; Schoenfeld, 1992
- Overlap between proving and problem solving
  - Furinghetti and Morselli, 2009; Weber, 2005
- "The approach that an individual takes to constructing a proof will influence what learning opportunities are afforded by the proof production" (Weber, 2005, p. 352).

Multi-Dimensional Framework

- Four phases of problem solving:
  - Orienting: Sense-making, organizing, constructing
  - Planning: Conjecturing, Imagining, and evaluating
  - Executing: Calculating, computing, constructing
  - Checking: Verifying, decision-making
- Carlson and Bloom noted that the last three phases were cyclic, meaning that if checking lead to an incorrect solution, the process would start back to planning

Multi-Dimensional Framework, cont.

- There are also 4 attributes that crossed through these four phases:
  - Resources: conceptual knowledge of the topic
  - Heuristics: algorithms, pictures, shortcuts, experience-based techniques
  - Affect: emotion, motivation, interest in problem
  - Monitoring: Figuring out if something is working, feel of the situation

Multi-Dimensional Framework, cont.

Phase • Behavior	Resources	Heuristics	Affect	Monitoring
Orienting • Sense making • Organizing • Constructing	Mathematical concepts, facts and algorithms were accessed when attempting to make sense of the problem. The solver also scanned her/his knowledge base to categorize the problem.	The solver often drew pictures, labeled unknowns and classified the problem. (Solvers were sometimes observed saying, "this is an X kind of problem.")	Motivation to make sense of the problem was influenced by their strong curiosity and high interest. High confidence was consistently exhibited, as was strong mathematical integrity.	Self-talk and reflective behaviors helped to keep their minds engaged. The solvers were observed asking, "What does this mean?"; "How should I represent this?"; "What does that look like?"
Planning Conjecturing Imagining Evaluating	Conceptual knowledge and facts were accessed to construct conjectures and make informed decisions about strategies and approaches.	Specific computational heuristics and geometric relationships were accessed and considered when determining a solution approach.	Beliefs about the methods of mathematics and one's abilities influenced the conjectures and decisions. Signs of intimacy, anxiety, and frustration were also displayed.	Solvers reflected on the effectiveness of their strategies and plans. They frequently asked themselves questions such as, "Will this take me where I want to go?", "How efficient will Approach X be?"
Executing • Computing • Constructing	Conceptual knowledge, facts and algorithms were accessed when executing, computing and constructing. Without conceptual knowledge, monitoring of constructions was misguided.	Fluency with a wide repertoire of heuristics, algorithms, and computational approaches were needed for the efficient execution of a solution.	Intimacy with the problem, integrity in constructions, frustration, joy, defense mechanisms and concern for aesthetic solutions emerged in the context of constructing and computing.	Conceptual understandings and numerical intuitions were employed to reflect on the sensibility of the solution progress and products when constructing solution statements.
Checking • Verifying • Decision making	Resources, including well-connected conceptual knowledge informed the solver as to the reasonableness or correctness of the solution attained.	Computational and algorithmic shortcuts were used to verify the correctness of the answers and to ascertain the reasonableness of the computations.	As with the other phases, many affective behaviors were displayed. It is at this phase that frustration sometimes overwhelmed the solver.	Reflections on the efficiency, correctness and aesthetic quality of the solution provided useful feedback to the solver

From Carlson and Bloom (2005), p. 67

Data Collection

- The data is from one professor, Dr. G and a graduate student, L.
- The data was collected using a LiveScribe pen.
- All the data was transcribed so that both the writing and speaking syncs with the time.
- The notes were the semigroup portion of the "Understanding and Constructing Proofs" notes, a Modified Moore Method course (Savic, 2011; Selden, McKee and Selden, 2010)

Data Collection (cont.)

- I transcribed both participants' proving processes on Theorem 20: "If S is a commutative semigroup with no proper ideals, then S is a group."
- I also asked two other individuals to code three different sections and we all agreed on a set of "rules" of coding.

Coding rules

- Planning
  - All breaks (although a participant may not be conscious of it)
  - All pauses that pre-empted the “executing” phase
- Executing
  - Writing that was part of the proof
  - Speaking was never considered
- Checking
  - Any “crossing out” or elimination of any part of the proof that had been “executed”

Example 1

Time	Writing	Coding
7:02 AM	Th 20: A comm semigp w/ no proper ideals is a gp.	Orienting (Resources)
7:03 AM	Hmm . . . I'm taking a break, breakfast, etc. Back to this later. Must think on this.	Orienting (Resources)
BREAK 7:04 AM - 8:07 AM	Planning	
8:07 AM	Ok, I thought about this while on a cold walk in the fog. Pf: Given $g \in S$, a semigp., consider the ideal g. . . . (Then he stops and puts "comm." between "a" and "semigp.") . . . $gS = \{gs \mid s \in S\}$. Since S has no proper ideals, $gS = S$, so $\exists g^{-1} \in S \ni gg^{-1} =$ (32 second pause, then he strikes through the whole proof)	Planning (Heuristics) Executing (Resources) Checking (Monitoring) Executing (Resources) Checking (Monitoring)
8:09 AM	First need an identity, not given. (Then he goes back to the expression "$gg^{-1} =$" and writes a question mark with a circle around it.) Turn page.	Checking (Resources), Planning (Resources) Cycling back (Back to planning, re-approaching to answer why there is no identity)

Example 1 Analysis

- 7:02 AM-7:04 AM
  - Orienting: organizing, sense-making. Dr. G is a professor, and the theorem is stated quite easily, hence the quickness of the orientation
- 7:04 AM-8:07 AM
  - Planning: note that I do not what happened in that time, but I do know that Dr. G returned from his cold walk with an idea to execute.
- 8:07 AM-8:09 AM
  - Executing: constructing and calculating.
- 8:09 AM:
  - Checking: Noticing that he needs an assumption (an identity), Dr. G has to go back and plan more on how to approach the proof.

Instance of the Conjecturing-Imagining-Evaluating Cycle

Time	Speaking	Coding
9:51 AM	But ok, if S is a commutative semigroup with a minimal ideal, then it's a group. Let's see.	Orienting (Resources)
	If it has a zero element, then that will be a minimal ideal.	Planning (Resources)
[Cycle Starts Here]	Does that make it a group?	Planning (Conjecturing)
	(Silence for 13 seconds)	Planning (Imagining)
	Well no,	Planning (Evaluating)
	what about the non-negative integers?	Planning (Conjecturing)

Instance of the Planning-Executing-Checking cycle/Multiple Codes

Time	Writing	Coding
10:19 AM	First we want to show S has an identity 1. (pauses for 45 sec)	Planning (Heuristics) Planning (Cycling)
10:20 AM	(pauses for 20 sec) If possible. Suppose S has no identity. Then for every $a \in S, ab \neq a$ for all	 Planning (Monitoring) Executing (Heuristics) Executing (Resources)
10:21 AM	(pauses for 25 sec, then crosses out “Then for every $a \in S, ab \neq a$ for all”) Let $a \in S$. Let $A = \{ab: b \in S, ab \neq a\}$.	Checking (Monitoring), Planning Executing (Resources)

Where the framework differs from the observed proving process

- Re-orienting his approach to the proof
  - Dr. G was “suspicious that the theorem was true,” and then seeking a counter-example. He states that he looked up the definition of “ideal”
- Impasses and incubation
  - Dr. G took several extended breaks during the proving process of Theorem 20.
- Planning and Checking
  - L demonstrated that he planned and checked multiple times during the proving process

Further expansion of phases

- Planning – Local and global
  - Approaching a proof with a certain proof framework (Selden and Selden, 1995) vs. proceeding on a small part of a proof
- Checking – Local and global
  - Checking to see if a proof is sound vs. checking to see if there are some minor errors

Conclusion

- These snippets of the overall data are rich with the same notions Carlson and Bloom (2005) saw in their examination of problem solving, suggesting that problem solving and the proving process may share many properties.
- But the coding also revealed many differences, including multiple categories for a certain instance.
- There seemed to also be a proposed refinement for some categories.

Limitations

- Unlike Carlson and Bloom (2005), I could not be in the room to view any affect or certain heuristics.
- I also did not encourage talking while proving to the participants, instead seeking a naturalistic setting for the provers.
- There is a certain amount of conjecture in assigning phases to breaks.

A proving process activity

- Students could do their homework or a test with a LiveScribe pen
- Their data could then be made into a movie or Pencast PDF for the student
- A homework assignment would then be to code their proving process using either a proving-process framework or Carlson and Bloom’s framework
- The idea would be to make explicit the proving process early in a student’s mathematical career

References

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Questions to discuss

- Are there any other problem-solving or proving process frameworks that would be helpful in this investigation?
- Would it be beneficial to ”break down” the anatomy of a proof or the proving process? Would it be more beneficial to modify Carlson and Bloom’s framework for proving?