Developing Reinvention Materials in Ring Theory: Analysis of Students' Mathematical Activity

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Introduction

Outline:

- Background and literature review
- Research questions
- Methodology: student population and building the notes
- Results: general student behaviors with detailed from a sub-unit
- Future research

Background: context

I got to teach Abstract Algebra! I identified several learning objectives and goals:

- students develop the habits mathematical inquiry as novice mathematicians
- students internalize and use the "categorical" themes from exploration of structure in the discipline
- students connect their understanding of abstract algebraic structure with previous learning; pre-service teachers value this perspective
- students build conceptions of groups and rings including Cancellation and quotient rings

Course Design Idea

The course is structured around an **inquiry-based** *development of groups and then* **guided reinvention** *of rings.*

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Background: guided reinvention

- RME;
- guided reinvention

Background: guided reinvention and algebra

- Dubinsky et al.;
- Larsen et al.;
- Cook;
- Sfard - dual nature;
- Simpson & Stehlikova - apprehending structure

Research Questions

SS^+ - elements, types of elements, rings, types of rings, multi-level properties

Research Questions

- *Can we develop materials that acknowledge the phases of apprehension of structure and that support students in developing a conception of Cancellation?*
- *How does student thinking evolve in response to these materials?*
- *How might we modify these materials for future iterations?*

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Methodology: student population

- institution: midwestern, religiously affiliated, small, liberal arts college
- students: 12 students (6m/6f); juniors/seniors; Mathematics and/or Secondary MathEd majors
- course: 300-level course; required for these degrees; 2^+ previous proof-based courses (Discrete & Linear Algebra); 10-week trimester course on groups and rings
- instruction: taught by one of the researchers; using IBL and guided reinvention approaches; students prepare ideas for class and we explore/discuss/present/polish/extend

Methodology: building the notes

- some ideas inspired by previous research (Larsen, Cook)
- some drafted by Cook for another course ahead of this schedule
- some are scaffolded (the example here); others more open (try to transfer from groups... go!)
- new tasks designed in response to the concepts that were hard for students previously or to have them encounter phenomena they hadn't or wouldn't notice otherwise
- discuss/share drafted tasks especially Cook/Katz; brainstorm about challenging concepts and ways to scaffold
- grounded analysis of the video data

Results: general

The students successfully built a tremendous amount of ring theory content:

- definition of rings, commutative, division rings, fields
- units, zero-divisors, Cancellation, strong closure
- some basic arithmetic properties

The students showed strong "categorical transfer" from groups:

- subrings, product rings, quotient rings
- homomorphisms (isomorphism equivalence relation)
- inherited properties (commutativity)
- *cyclic rings

Results: general

The students behaved like novice mathematicians:

- generated definitions, conjectures, ideas
- planned and executed example-based exploration
- powerful behavioral norms
- uncomfortable, stressed

The students started to reframe previous knowledge:

- generalization of Linear Algebra
- Cancellation in particular connected with high school concepts
  $0 = x^2 - 1 = (x+1)(x-1)$
  $\Rightarrow 0 = x + 1$ OR $0 = x - 1$

Results: detailed

Day 1 activity $\rightarrow$ Cancellation

Instructional Sequence: Days 5&6

- *Classify equations of the form $a + x = b$ and $a * x = b$ with a, b in various rings ($\mathbb{R}, \mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$) based on the number of solutions.*
- *Develop a classification of the types of elements (a) based on solution-set behavior (in the multiplicative equation).*
- *Which kinds of elements do these rings ($\mathbb{R}, \mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}, 2\mathbb{Z}, M_2(\mathbb{R})$, others) contain? Do these rings have Cancellation?*
- *Articulate a conjecture about the relationship between types of elements and Cancellation. Prove that conjecture.*

Results: detailed

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Results: detailed

Solving $a * x = b$: Zero-divisors (Day 5)

- They read off computations from $\mathbb{Z}_{12}$ Cayley table after recalling that it was "weird".
- They error correct using patterns from the table.
- There are some typos, and they clarify with computations in $\mathbb{Z}_{12}$. (HDV455 @ 0:42)
- They attend to factoring properties rather than the new phenomenon.
- The students struggle to talk about or even name these elements.

Results: detailed

Solving $a * x = b$: Zero-divisors (Day 6)

- They immediately observe that Cancellation fails in $\mathbb{Z}_{12}$ because of multiple solutions.
- They observe that zero-divisors were the only ones that gave multiple solutions, so it makes sense that they are related to Cancellation failing.
- They make a claim about Cancellation failing for $\mathbb{Z}_n$ (n composite), though they return to examples rather than general properties as needed.
- They easily make the conjecture: Cancellation IFF no zero-divisors.
- When they summarize their chart findings, they are much more slick and use language of types of elements. (MOV002 @ 49:23)
- Together, we can move to a general method of using a zero-divisor to get Cancellation to fail.

Results: detailed

Solving $a * x = b$: Units (Days 5&6)

- They start by grouping elements with inverses.
- They justify unique solutions using the inverse property (which aligns with the proof from groups).
- They combine cases using the awareness that only unit properties matter.
- They talk about units functioning the same in all rings.
- They are uncomfortable with Cancellation working outside the case in which units are present to power the proof they know.
- They embed $\mathbb{Z}$ in $\mathbb{R}$ to make elements into units and conclude that the equations have 0-1 solutions!
- They discuss Cancellation in group-rings, for which all non-zero elements are units!

Future Research

Modify the materials:

- help students decide to attend to a or to b (decision tree?)
- help students develop solution types: none, unique, *some*, all
- help students classify types of elements (a) and name them meaningfully
- consider using language of injective/surjective/bijective or systems of equations to connect this work
- support them as they work on general zero-divisor issues

Other questions:

- Are there other content domains in which these ideas are salient?
- What is the genetic decomposition of student difficulties? How does this depend on the groups-rings sequencing?

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References

Cook;
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Larsen et al.;
Sfard;
Simpson & Stehlikova;

Conclusion

Thank you!

Questions?
Suggestions for writing about this data or for collecting more data?

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