What do Mathematicians do When They Reach a Proving Impasse?

RUME Conference

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February 24th, 2012

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Motivation and Questions

- During my last research project on logic there were 3 students who should have been able to prove a theorem but could not in a 45 minute interview.
- All 3 "got stuck" during the interview.
- How can people be observed constructing proofs alone (with unlimited time)?
- Do mathematicians "get stuck" and how do they get "un-stuck?"

Background Literature

- Mathematicians' knowledge
  - Actions during proof validations (Weber, 2008)
  - Mathematicians' learning (Burton, 1999; Wilkerson-Jerde & Wilensky, 2011)
  - Using diagrams to construct proofs (Samkoff, Lai, & Weber, 2011)
- Students' proving
  - Difficulties (Moore, 1994; Weber & Alcock, 2004)
  - Validations of proofs (Selden & Selden, 2003)
  - Comprehension of proofs (Conradie & Frith, 2000; Mejia-Ramos, et al., 2010)

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Impasses

- Impasse – A period of time when a prover feels or recognizes the argument is not progressing and he or she has no new ideas
  - Also known as "getting stuck" or "spinning one's wheels"
  - Different from an impasse defined for automated computer provers (Meier & Melis, 2005)
- Two kinds of actions to recover from an impasse
  - Directly relating to the ongoing argument
  - Doing something unrelated which could be mathematical or non-mathematical

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Incubation

- Incubation – a period of time, following a proof attempt, during which similar activity does not occur
- The second stage of the 4 stages of creativity (Wallas, 1926)
  - Preparation, Incubation, Illumination, Verification
- Poincare, Hadamard, and other mathematicians have described a period of incubation, followed by an "insight"
- Apparently should have interest in finding the solution for incubation to have any effect

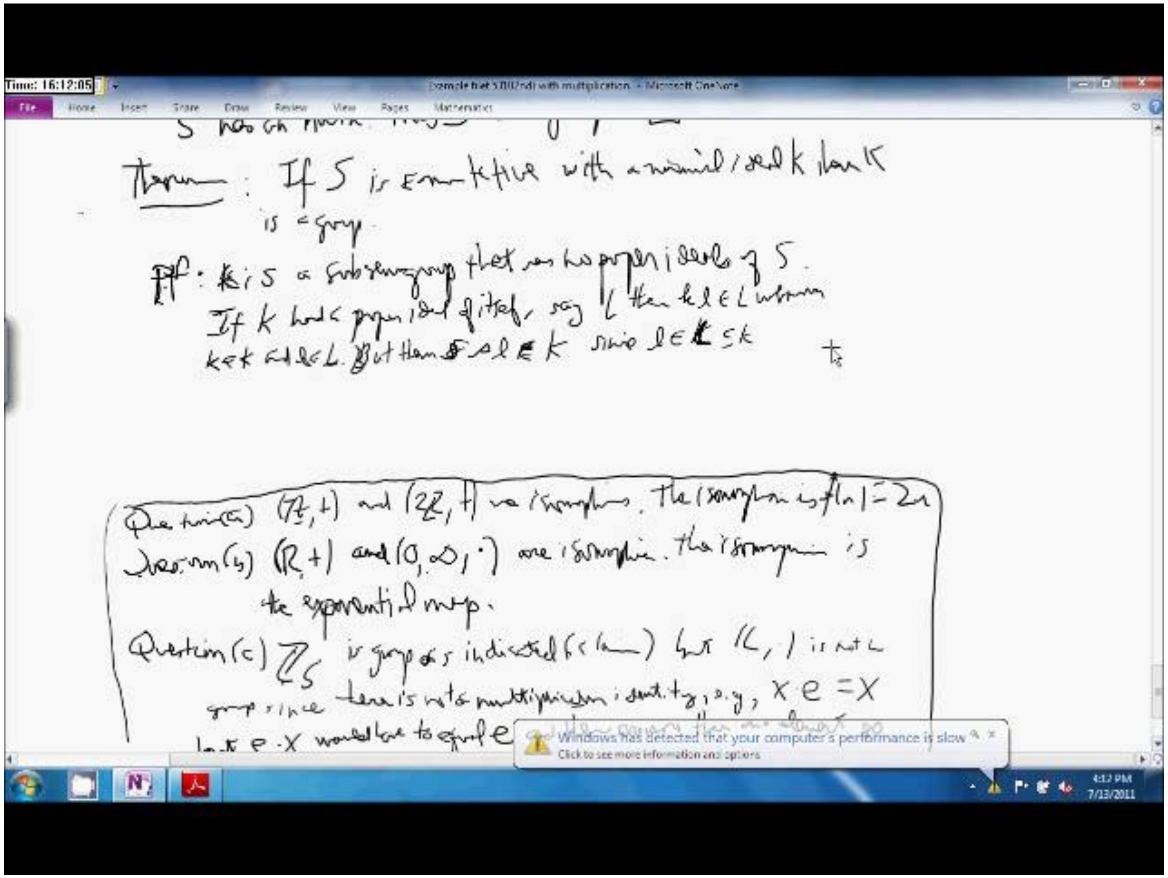
Participants and Tasks

- Nine research mathematicians (3 algebraists, 2 analysts, 3 topologists, 1 logician)
- Tasks – prove theorems in notes on semigroups (10 definitions, 13 theorems, 7 example requests, and 4 questions)
- Chosen for two reasons
  - Material (I hoped) was unfamiliar but accessible
  - Last two theorems require non-obvious lemmas and were difficult for students

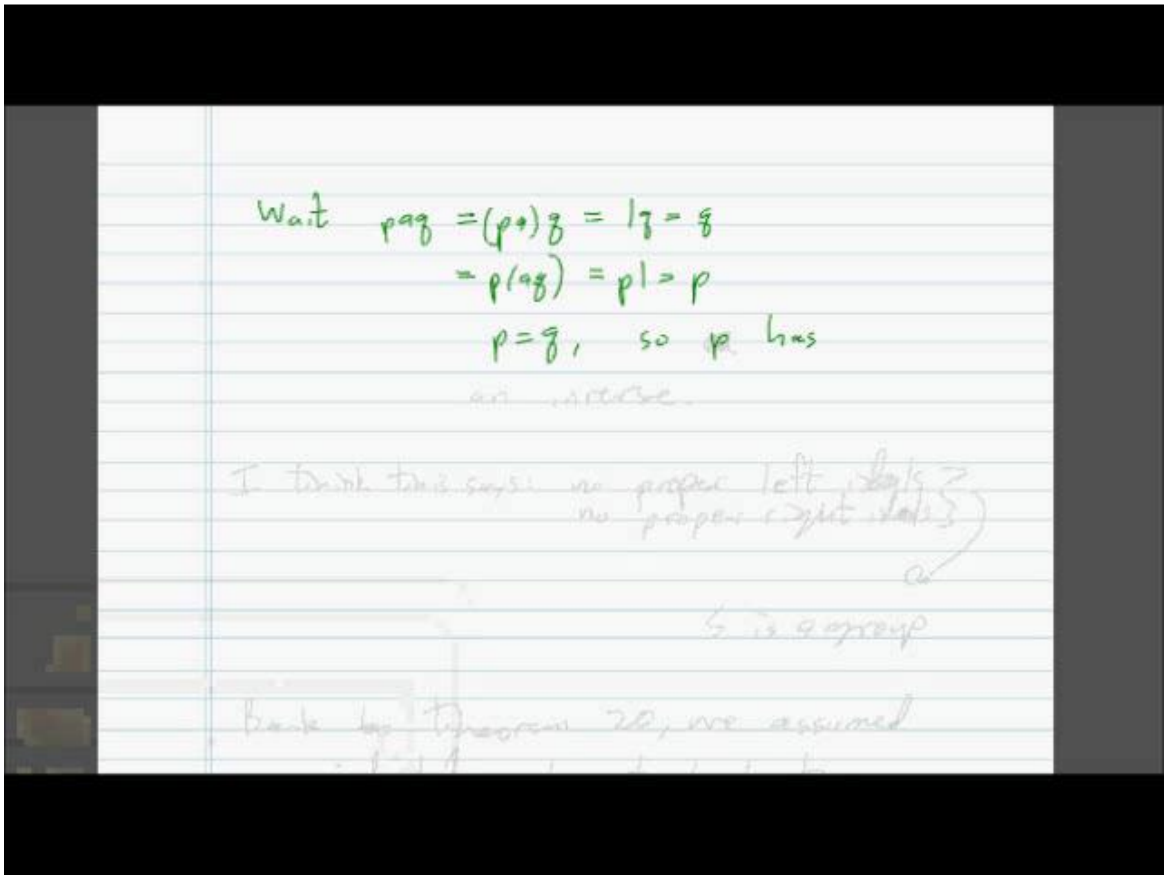
Data Collection

- Electronically:
  - The first four mathematicians proved on a tablet PC, set-up with CamStudio (screen-capturing software) and OneNote (space for their writing).
  - The final five mathematicians proved with a LiveScribe pen and special paper, capable of recording audio and writing in real-time.
- Both had date and time stamps for each writing session
- Advantages:
  - Used at the participant's leisure
  - Real-time recording of the proving process
  - Never done before

Example of Tablet PC



Example of LiveScribe pen



Data Collection, cont.

- Each mathematician kept the equipment for 2-7 days.
- I analyzed the screen captures and the proof attempts.
- One or two days later, I interviewed the mathematicians about the proofs and the proving attempts.
- I also had two videoed "focus group" sessions: one for the tablet participants, the other for the LiveScribe pen participants.
- Two mathematicians volunteered the choice of semigroups was judicious:
  - Grasp concepts quickly
  - At least one of the theorems was challenging to prove

Summary Data

- 4 of the 9 professors had problems with the equipment, and thus did not have "live" data
- 6 of the 9 professors had impasses with one of the last two theorems
- Average time of a professor's work on the technology: 2 hours, 5 minutes
- Average time of work overall: 19 hours, 56 minutes
- Average amount of pages written: Around 13

Dr. A

- Applied analyst
- Encountered impasse with the final theorem in the notes: "If S is a commutative semigroup with minimal ideal K, then K is a group."
- Done on a tablet PC
- Total time: 22 hours, 17 minutes (July 13, 2:44 PM - July 14, 1:01 PM)

Proving Process of Dr. A, Day 1

- 3:48 PM Attempted the proof by contradiction
- 3:54 PM Moved on to the final part of the notes containing a request for examples
- 4:05 PM Scrolls on the screen back up to view his first proof attempt, which he erased.
- 4:12 PM Attempted the proof again

Proving Process of Dr. A, Day 2

- Next screen capture at 11:07 AM of Day 2.
  - Using mappings and inverse mappings of elements
- "I don't know how to prove that K itself is a group."
- After a 30-minute gap, he proved the theorem successfully

Dr. A’s Exit Interview

- Dr. A acknowledged his impasse:
  - “One has to show there aren't any sub-ideals of the minimal ideal itself, considered as a semigroup, and that's where I got a little bit stuck.”
- Dr. A gets out of this impasse (consciously) by walking around and doing his departmental duties.

Dr. B

- Algebraist
- Encountered impasse with the penultimate theorem in the notes: "If S is a commutative semigroup with no proper ideals, then S is a group."
- Done on a tablet PC
- Did not get any screen captures due to failure with software
- Total time: 5 hours, 40 minutes (August 3, 7:25 AM – August 3, 1:05 PM)

Proving Process of Dr. B

- Dr. B wrote: "Stuck on [theorem] 20. It seems you need $1 \in S$ [in the hypothesis], but I can't find a counterexample to show this."
- Moved on to the next theorem, which he proved correctly, but then crossed it out.
- Then Dr. B went to the final question dealing with examples of isomorphisms of semigroups.
- Dr. B was interrupted to go to lunch.
- After lunch, Dr. B proved both theorems correctly.

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Dr. B's Exit Interview

- Dr. B stated that he had created a property that had confused him, and thought that he needed to assume that there was an identity.
  - "I probably spent 30 minutes to an hour trying to come up with a crazy example."
- He said he got out of his impasse by going to lunch with his family, noting that he would have worked on the problem continuously if not for the lunch.

Actions to Overcome Impasses

- Viewing the impasses, the action to overcome relates directly or does not relate to the impasse
- Hence, the actions are separated into two categories:
  - Directly related actions
  - Seemingly unrelated actions
- All the actions to overcome impasses are accompanied by exit interview quotes from all 9 professors supporting the action

Directly Related Actions

- Using methods that occurred earlier in the session
  - "It would be fairly easy to prove…it's likely an argument, kind of like the one I already used…" (Dr. H)
- Using prior knowledge from their own research
  - "I'm trying to think if there's anything in the work that I do that...I mean some of the stuff I've done about subspaces of $L^2(\mathbb{R})$, umm...there are things called principal shift invariance spaces that the word principal comes into play." (Dr. A)

Directly Related Actions, cont.

- Using a database of proving techniques
  - "Your brain is randomly running through arguments you've seen in the past… standard techniques that keep running through my head, sort of like downloading a whole bunch at the same time and figuring out which way to go." (Dr. F)
- Doing other problems and coming back to their impasse
  - "I moved on because I was stuck…maybe I was going to use one of those examples…I might get more information by going ahead." (Dr. B)

Unrelated Actions

- Doing other mathematics
  - "What I try to do is to keep three projects going…I make them in different areas and different difficulty levels…" (Dr. E)
- Walking
  - "When I'm stuck, I often feel like taking a break. And indeed, you come back later and certainly for a mathematician you go off on a walk and you think about it." (Dr. G)
- Doing tasks unrelated to mathematics
  - "Yeah I'll do something else, and I'll just do it, and if there's a spot where I get stuck or something, I'll put it down and I'll watch TV, I'll watch the football game, or whatever it is, and then at the commercial I'll think about it and say yeah that'll work…" (Dr. E)

Unrelated Actions, cont.

- Going to lunch/eating
  - "So I had spent probably the last 30 min to an hour on that time period working on number 20 going in the wrong direction. Ok, so I went to lunch, came back, and while I was at lunch, I wasn't writing or doing things, but I was just standing in line somewhere and it occurred to me the…(laughs)…how to solve the problem." (Dr. B)
- Waking up
  - "It often comes to me in the shower…you know you wake up, and your brain starts working and somehow it just comes to me. I've definitely gotten a lot of ideas just waking up and saying "That's how I'm going to do this problem." (Dr. F)

Why is incubation important?

- Dr. G, from the focus group session: "When we are working on something, we are usually scribbling down on paper. When you go take a break,… you are thinking about it in your head without any visual aides….[walking around] forces me to think about it from a different point of view, and try different ways of thinking about it, often global, structural points of view."
- Dr. F: "You just come back with a fresh mind. You're zoomed in too much and you can't see anything around it anymore."
- Dr. A: "I do have a belief that if I walk away from something and come back it's more likely that I'll have an idea than if I just sit there."

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Other Observations

- Two of the mathematicians misread the last theorem, "If S is a commutative semigroup, and K is a minimal ideal, then K is a group."
  - Both emailed me after I had mentioned the misreading and had sketched a proof in the email
- 3 professors looked for counter-examples of some theorems, noting that at first they seemed to be false claims

Future Research

- Compare the graduate students' data to the mathematicians in the proving process
- Using the data collection technique to help correct students' proving approaches
  - Akin to sports' "film sessions"
  - May be helpful in transition-to-proof or proof-based courses

References

- Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37*, 121-143.
- Conradie, J., & Frith, J. (2000). Comprehension tests in mathematics. *Educational Studies in Mathematics, 42*, 225-235.
- Hadamard, J. (1945) *The Mathematician's Mind*. Princeton University Press.
- Krashen, S. (2001). Incubation: A neglected aspect of the composing process? *ESL Journal, 4*, 10-11.
- Meier, A., & Melis, E. (2006). Impasse-driven reasoning in proof planning. In M. Kohlhase (Ed.), *Mathematical Knowledge Management: 4th International Conference MKM 2005* (pp. 143-158). Berlin: Springer-Verlag.
- Mejia-Ramos, J. P., Weber, K., Fuller, E., Samkoff, A., Search, R., & Rhoads, K. (2010). Modeling the comprehension of proof in undergraduate mathematics. *Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 1-22). Raleigh, NC.: Available online.
- Moore, R. (1994). Making the transition to formal proof. *Educational Studies in Mathematics 27*, 249-266.

References, cont.

- Samkoff, A., Lai, Y., & Weber, K. (2011). How mathematicians use diagrams to construct proofs. *Proceedings of the 14th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 430-444). Portland, OR.: Available Online.

- Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education, 34*, 4-36.

- Sio, U. N., & Ormerod, T. C. (2009). Does incubation enhance problem solving? A meta-analytic review. *Psychological Bulletin, 35*, 94-120.

- Wallas, G. (1926) *The Art of Thought*. New York: Harcourt, Brace and Company.

- Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education, 30*, 431-459.

- Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*, 209-234.

- Wilkerson-Jerde, M. H., & Wilensky, U. J. (2011). How do mathematicians learn math? Resources and acts for constructing and understanding mathematics. *Educational Studies in Mathematics, 78*, 21-43.

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Questions?

- How can we gain information on when and how incubation is used in mathematics?
- Is it important to let students know about incubation?
- How can we collect all actions that mathematicians use to recover from impasses?
- Also, can we encourage students to take some of these actions to recover from impasses?
- Acknowledgments: Thanks to all the professors and graduate students who served as participants in this research.
- If you have further questions, please contact me at milos@nmsu.edu. Thank you!

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