What do mathematicians do when they reach a proving impasse?

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I report how two mathematicians came to impasses while constructing proofs on an unfamiliar topic, from a set of notes, alone, and with unlimited time. By an impasse, I mean a period of time during the proving process when a prover feels or recognizes that his or her argument has not been progressing fruitfully and that he or she has no new ideas. What matters is not the length of time but its significance to the prover and his or her awareness thereof. I point out two kinds of actions these mathematicians took to recover from their impasses: one relates directly relates to the ongoing argument, while the other consists of doing something unrelated to the ongoing argument which can be mathematical or non-mathematical. Data were collected using a new technique being developed to capture individuals' autonomous proof constructions on tablet computers in real-time.

Key words: university level, proof, mathematicians, impasse, tablet PC

This preliminary report presents part of an ongoing larger study of mathematicians and graduate students constructing proofs on an unfamiliar topic, from a set of notes, alone, and with unlimited time. During separate data collection sessions, each of two mathematicians came to an impasse in proving certain theorems. This study investigated what actions these mathematicians took to try to recover from those impasses. Data were collected in real-time using a new technique being developed to capture individuals' autonomous proof constructions on tablet computers.

BACKGROUND LITERATURE

While there has been research on mathematicians' actions during proof validation (Weber, 2008), on how mathematicians learn new mathematics (Burton, 1999; Wilkerson-Jerde & Wilensky, 2011), and on how mathematicians use diagrams to construct proofs (Samkoff, Lai, & Weber, 2011), to date there appears to have been little or no research on what mathematicians do when they reach an impasse during proving. This may have implications for helping students with proving.

To date, research on university students' proving has been concerned with a variety of topics including: difficulties they encounter during the proving process (Moore, 1994; Weber & Alcock, 2004), with their validations of proofs (Selden & Selden, 2003), and with their comprehension of proofs (Conradie & Frith, 2000; Mejia-Ramos, et al., 2010). Such research is helpful in teaching proving. In the same way, it would be interesting to know what students do when they are actually in the process of proving, and in particular, what they do when they come to an impasse. This study is a start in that direction.

THEORETICAL FRAMEWORK

By an *impasse*, I mean a period of time during the proving process when a prover feels or recognizes that his or her argument has not been progressing fruitfully and that he or she has no new ideas. What matters is not the length of time but the significance to the prover and his or her awareness thereof. Mathematicians themselves often colloquially refer to impasses as "being stuck" or "spinning one's wheels." This is different from simply "changing directions," when a prover decides, without much hesitation, to use a different method, strategy, or key idea. I will point out two kinds of mental or physical actions a prover may take to recover from an impasse. One kind of action directly relates to the ongoing argument. The other kind of action is doing something else unrelated to the ongoing argument which can be mathematical or non-mathematical. Examples of both will be provided.

Computer scientists working on automatic theorem provers have considered how machines overcome impasses, noting that "when an expected progress does not occur or when the proof process gets stuck, then an intelligent problem solver analyzes the failure and attempts a new strategy" (Meier & Melis, 2006). However, this is different from my description of an impasse because it does not have a time component and for a person, analyzing the failure, can be considered as a continuation of the proving process.

DATA COLLECTION TECHNIQUE

Several mathematicians agreed to participate in this study. They were provided with notes on semigroups containing definitions, requests for examples, and theorems to prove. The notes were a modified version of the semigroups portion of the notes for a Modified Moore Method course for beginning graduate students. This topic was selected because the mathematicians would find the material easily accessible, and because there are two theorems towards the end of the notes that have caused substantial problems for beginning graduate students.

Data on the mathematicians' written work, with time-stamps, were collected electronically on a tablet PC. I explained how to use the stylus that came with the computer, the CamStudio screen recording software, and Microsoft OneNote, which was the space in which the mathematicians wrote their proof attempts. All mathematicians, including the two described here, kept the tablet PC for 2-4 days. After the computer was returned, I analyzed the screen captures (like small movies in real time) and the mathematicians' proof writing attempts. One or two days later, I conducted an interview during which I asked each mathematician about his proofs and proof-writing. The two mathematicians offered that the choice of semigroups was judicious, because they were able to grasp the definitions and concepts quickly, and because at least one of the theorems had been somewhat challenging to prove.

WHAT THE MATHEMATICIANS DID

In this paper, I focus on just two mathematicians: Dr. A, an applied analyst, and Dr. B, an algebraist.

In his proofs, Dr. A encountered an impasse on the final theorem in the notes: "If S is a commutative semigroup with minimal ideal K, then K is a group." He first attempted a proof by contradiction. After two and a half minutes, he moved on to the final part of the notes containing a request for examples, which he provided quite quickly. Dr. A then spent another 8 minutes, during which time he scrolled up on the OneNote program in order to view his first contradiction proof attempt, which he then erased. He then unsuccessfully attempted another proof, trying to utilize his previous correct proof of the penultimate theorem. The screen-capturing of these unsuccessful proof attempts started at 3:48 PM, and the session ended at 4:17 PM. The next screen capture started the following day at 11:07 AM, with Dr. A again attempting the proof, this time using mappings and inverse mappings of elements. Then he wrote, "I don't know how to prove that *K* itself is a group," thereby acknowledging that he was at an impasse. After that, there is a 30-minute gap between screen-captures until he finally proves the theorem successfully.

Dr. A indicated in his post-interview where he had had an impasse, noting "One has to show there aren't any sub-ideals of the minimal ideal itself, considered as a semigroup, and that's where I got a little bit stuck." Dr. A also indicated generally how he consciously recovers from impasses: he prefers to get "un-stuck" by walking around, but distractions caused by his departmental duties also help.

Dr. B got to an impasse on the penultimate theorem, "If *S* is a commutative semigroup with no proper ideals, then *S* is a group." Unfortunately with Dr. B, I did not get any screen captures, but his written proof attempts are very detailed and the exit interview was very informative. He wrote, "Stuck on [theorem] 20. It seems you need $1 \in S$, but I can't find a counterexample to show this." Dr. B then moved on to the final theorem on which Dr. A had had an impasse, proved it correctly, and then crossed his proof out. He then moved on to the final request for examples, explaining in his exit interview, "I moved on because I was stuck...maybe I was going to use one of those examples...I might get more information by going ahead." Dr. B's next approach was to create counterexamples. After considering his counterexamples for some time and taking his family to lunch, Dr. B proved both theorems correctly.

In the exit interview, Dr. B stated that he had created a property that had confused him, and thought that he needed to assume that there was an identity. Also, he said, "I probably spent 30 minutes to an hour trying to come up with a crazy example. I went to lunch and while I was at lunch, then it occurred to me that I was thinking about it the wrong way. So I went back then and it was quick."

RESULTS

The actions directly-related to the ongoing argument that these two mathematicians took to recover from impasses were: utilizing semigroup proof techniques that they had used earlier in the sessions, utilizing prior knowledge from their own research areas, and generating examples and counterexamples. The second kind of action involves doing something else. In the data, these were doing subsequent problems in the notes and coming back to their unfinished proof attempts, and engaging in other “non-proof” activities (such as walking around the office, doing other tasks, going to lunch). The first one of these is mathematical, whereas the remaining are non-mathematical diversions. Most of these actions were more or less automatic and not consciously noted by the mathematicians either during the session or in the exit interviews. In analyzing an action, it is sometimes difficult to distinguish between a conscious intention to recover from an impasse or a serendipitous action later recognized as having been helpful.

Doing other activities and coming back to an unfinished problem might be considered an example of *incubation*, which is the process by which the mind goes about solving a problem subconsciously and automatically, and which happens best when one takes a break from creative work (Krashen, 2001). While there are many reports of experiments on incubation in the psychology literature (Sio & Ormerod, 2009), they typically allow only a short time for incubation. However, both mathematicians stated that when they received the notes, they immediately glanced at them to estimate how long the proofs might take, but both started proving the next day. It is difficult to know whether there was an incubation effect due to actually commencing their proving the next day. How can we gain information on when and how incubation is used in mathematics? Is it important to let students know about incubation? How can we collect all actions that mathematicians use to recover from impasses? Also, can we encourage students to take some of these actions to recover from impasses?

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