Where is the Logic in Homology Proofs?

Milos Savic, Department of Mathematics, New Mexico State University

Why is this an interesting question?

Researchers in Mathematics Education have been split about the decision to teach an explicit unit in formal logic (predicate and propositional calculus) (Epp, 2003; Hanna & de Villiers, 2008). The aim of the analysis is to find the logic in proofs so that the question of how it should be taught can be better understood. If formal logic occurs quite a bit, then teaching a unit on predicate and propositional calculus might be a good idea; if formal logic is infrequent, then teaching logic in context, while teaching proving, might be more effective.

Motivation

This study was motivated by two factors. First, I had been coding proofs in a transition-to-proof course to determine any answers to the previous question, and the coding was happening concurrent to the Homology class I was taking. Secondly, it would be interesting to see results of the coding from both a course dealing with many topics versus a course dealing with one topic in depth.

Setting

The Homology course, which ran in the Fall 2010, consisted of topics such as Module Theory, Category Theory, Homology and Cohomology, with emphasis on Tor and Ext functors. The 10 proofs coded were from another student who had received perfect scores on those proofs.

Coding

In the iterative process of coding, I had used a "chunk-by-chunk" analysis. The "chunks" are similar to those used in analyzing short-term memory (Miller, 1956). They are small phrases that can be taken together as a "meaningful unit" in thinking. One iteration had two mathematics professors and I coding 6 theorems (4 at-home and 2 in group) to discuss the coding and categories.

Categories

In the process of chunking the transition-to-proof proofs, 13 categories emerged, and I used those same categories in my analysis of the Homology proofs. In this poster, I list 5 of the 13; the two that pertain to the question on logic, and the three that occurred most frequently in my analysis.

(**FL**) Formal logic: Any logic that is not common sense will be considered as "Formal Logic". This includes predicate and propositional calculus, and it is assumed that a beginning transition-to-proof student would not have this knowledge.

Example: In a proof "…If $x \in A$ and $x \notin B$, and $A, B \subseteq X$, then $x \notin (X - A) \cup B$…"

FL: "then $x \notin (X - A) \cup B$"

(**II**) Informal inference: "Informal inference" refers to an inference depending on common sense logic.

Example: In a proof: "…If $a \in A$ … $A \subseteq B$… then $a \in B$…"

II: "then $a \in B$"

(**A**) Assumption: We separate assumption into two sub-categories:

- Choice: When a symbol is chosen to represent an object (often fixed, but arbitrary) about which something will be proved – but not the assumption of additional properties given in a hypothesis
- Hypothesis: When the hypothesis of a theorem or argument is assumed (often stating properties of an object in the proof).

Example 1: For the theorem "For all $n \in \mathbb{N}$, if $n > 5$ then $n^2 > 25$."

A (Choice): "Let $n \in \mathbb{N}$" (Fixed but arbitrary)

A (Hypothesis): "Suppose $n > 5$"

(**DEF**) Definition of: When the proof-writer uses a definition of a mathematical object.

Example: "…so $x \in A$… thus $x \in B$… Then $x \in A \cap B$…"

DEF: "Then $x \in A \cap B$".

(**IR**) Interior reference: "Interior reference" refers to when a chunk of a proof calls on anything stated earlier in the proof.

Example: In a proof: "…Let $x \in A$ … $A \subseteq B$… Since $x \in A$, $x \in B$…"

IR: "Since $x \in A$"

Example

Theorem 2: Let R be a domain and I a nonzero ideal in R that is a free R-module; prove that I is a principal ideal.

Proof: First suppose that $\alpha: R \oplus R \to R$ is a nonzero homomorphism. Set $\alpha\big((1,0)\big) = s_1$ and $\alpha\big((0,1)\big) = s_2$, and let $I_1 = s_1 R$ and $I_2 = s_2 R$. For any $(r_1, r_2) \in R \oplus R, \alpha(r_1, r_2) = \alpha(r_1(1,0) + r_2(0,1)) = r_1\alpha\big((1,0)\big) + r_2\alpha\big((0,1)\big) = r_1 s_1 + r_2 s_2$, so $\ker\alpha = \{(r_1, r_2) \in R \oplus R | r_1 s_1 = -r_2 s_2\}$. Since $\alpha \neq 0$, we cannot have both $I_1 = 0$ or $I_2 = 0$. Suppose first that $I_1 = 0$ so that necessarily $I_2 \neq 0$; then in this case, $\ker\alpha = \{(r_1, r_2) \in R \oplus R | r_1 s_1 = -r_2 s_2\} = \{(r_1, r_2) \in R \oplus R | 0 = -r_2 s_2\} = \{(r_1, 0) \in R \oplus R\}$ since R is a domain. Similarly, if $I_2 = 0$ and $I_1 \neq 0$, the kernel of α consists of all the elements of the form $(0, r_2)$. Lastly, if both $I_1 \neq 0$ and $I_2 \neq 0$, then by 1, there is a nonzero $x \in I_1 \cap I_2$; say $x = t_1 s_1 = t_2 s_2$ for some $t_1, t_2 \in R - \{0\}$. The element $(-t_1, t_2)$ is then a nonzero element in the kernel of α since $\alpha\big((-t_1, t_2)\big) = -t_1 s_1 + t_2 s_2 = x - x = 0$. Thus in every case, α has a nontrivial kernel; hence there is no injection $R \oplus R \to R$. Now, as I is free, $I \cong \bigoplus_\Gamma R$ for some indexing set Γ; if $|\Gamma| \geq 2$, then there is an injection $R \oplus R \hookrightarrow I \subseteq R$ which is impossible. Therefore, $I \cong R$ showing that I is principal.

First suppose that $\alpha: R \oplus R \to R$ is a nonzero homomorphism.	Assumption (Choice)
Set $\alpha\big((1,0)\big) = s_1$	Relabeling
and $\alpha\big((0,1)\big) = s_2$,	Relabeling
and let $I_1 = s_1 R$	Assumption (Choice)
and $I_2 = s_2 R$.	Assumption (Choice)
For any $(r_1, r_2) \in R \oplus R, \alpha(r_1, r_2) = \alpha(r_1(1,0) + r_2(0,1))$	Definition of Scalar Multiplication
$= r_1\alpha\big((1,0)\big) + r_2\alpha\big((0,1)\big)$	Definition of Homomorphism
$= r_1 s_1 + r_2 s_2$	Interior Reference
, so $\ker\alpha = \{(r_1, r_2) \in R \oplus R | r_1 s_1 = -r_2 s_2\}$.	Definition of kernel, Algebra
Since $\alpha \neq 0$	Interior Reference
we cannot have both $I_1 = 0$ or $I_2 = 0$.	Formal logic
Suppose first that $I_1 = 0$	Assumption (Hypothesis)
…	…
Lastly, if both $I_1 \neq 0$ and $I_2 \neq 0$,	Assumption (Hypothesis)
then by 1, there is a nonzero $x \in I_1 \cap I_2$;	Exterior reference
say $x = t_1 s_1 = t_2 s_2$ for some $t_1, t_2 \in R - \{0\}$.	Relabeling
…	…
if $|\Gamma| \geq 2$,	Assumption (Hypothesis)
then there is an injection $R \oplus R \hookrightarrow I \subseteq R$	Informal inference
which is impossible.	Interior reference
Therefore, $I \cong R$ showing that I is principal.	Conclusion statement

Results

Formal logic only occurred once out of 170 chunks, where informal inference occurred 17 times. The top three categories were Definition, Assumption, and Interior Reference, which corresponded with the results from the other study.

	Chunks	A	ALG	C	D	DEF	ER	FL	II	IR	REL	SI	SIM
Totals	170	31	4	14	5	36	21	1	17	30	8	1	2
Percentage		18.2	2.4	8.2	2.9	21.2	12.4	0.6	10.0	17.7	4.7	0.6	1.2