The Mathematics Enthusiast

Volume 13
Number 3 *Number 3*

Article 6

8-2016

Mathematical Problem-Solving via Wallas' Four Stages of Creativity: Implications for the Undergraduate Classroom

Milos Savic

Mathematical Problem-Solving via Wallas' Four Stages of Creativity: Implications for the Undergraduate Classroom

Milos Savic[1]
University of Oklahoma

Abstract: The central theme in this article is that certain problem-solving frameworks (e.g., Polya, 1957; Carlson & Bloom, 2005) can be viewed within Wallas' four stages of mathematical creativity. The author attempts to justify the previous claim by breaking down each of Wallas' four components (preparation, incubation, illumination, verification) using both mathematical creativity and problem-solving/proving literature. Since creativity seems to be important in mathematics at the undergraduate level (Schumacher & Siegel, 2015), the author then outlines three observations about the lack of fostering mathematical creativity in the classroom. Finally, conclusions and future research are discussed, with emphasis on using technological advances such as Livescribe™ pens and neuroscience equipment to further reveal the mathematical creative process.

Keywords: mathematical creativity, problem solving, proving, fostering creativity, incubation, creative process

[1] savic@ou.edu

Introduction

Problem solving is a process that is declared important for mathematics and mathematics education. Schoenfeld (1992), after citing national studies by the National Council of Teachers of Mathematics and the National Research Council, stated that "there is general acceptance of the idea that the primary goal of mathematics instruction should be to have students become competent problem solvers" (p. 3). While mathematics education researchers continue to investigate problem solving to understand mechanisms that a solver goes through, many components of this problem-solving process built upon seminal work of Polya (1957) that provided four stages: (i) understanding the problem, (ii) developing a plan, (iii) carrying out the plan, and (iv) looking back. Polya's four-stage framework influenced other problem-solving frameworks, including Carlson and Bloom's (2005) *Multidimensional Problem-solving Framework*. However, the term "problem solving" has been defined in many ways, to the point where Chamberlin (2008) stated: "There is rarely an agreed upon definition of mathematical problem solving and reaching consensus on a conceptual definition would provide direction to subsequent research and curricular decisions" (p. 1).

Similarly, the term "mathematical creativity" has been defined in many ways, to the point where Mann (2006) stated: "An examination of the research that has attempted to define mathematical creativity found that the lack of an accepted definition for mathematical creativity has hindered research efforts" (p. 238). However, Mann (2006) also claimed that not investigating mathematical creativity to enhance students' efforts could "drive the creatively talented underground or, worse yet, cause them to give up the study of mathematics altogether" (p. 239). Mathematical creativity is either a process or a product (depending on the definition) that is declared important for mathematics and mathematics education. The Committee on

Undergraduate Programs in Mathematics (Schumacher & Siegel, 2015) stated that, “A successful major offers a program of courses to gradually and intentionally leads students from basic to advanced levels of critical and analytical thinking, while encouraging creativity and excitement about mathematics” (p. 9). According to Liljedahl (2009), “it is through mathematical creativity that we see the essence of what it means to ‘do’ and learn mathematics.” (p. 239). While mathematical creativity has been researched (e.g., Sriraman, 2004), many components of the creative process come from the psychologist Wallas (1926). Wallas stated that there are four stages of creativity: i) preparation (thoroughly understanding a problem), ii) incubation (when the mind goes about solving a problem subconsciously and/or automatically), iii) illumination (internally generating an idea after the incubation process, sometimes known as the AHA! experience), and iv) verification (determining whether that idea is correct). Mathematicians (Hadamard, 1945; Poincaré, 1946) have stated that they have experienced similar stages in their mathematical process, and primary and secondary school mathematics education researchers have used Wallas’ stages to explore problem solving (e.g., Prusak, 2015).

As discussed in the previous two paragraphs, there seems to be a connection between both problem solving and mathematical creativity, which evokes the following research question: are stages of “problem solving” and “mathematical creativity” equal sets, or is one a subset of another? In this article, the stance is that the stages of problem solving are a subset of mathematical creativity, applying Wallas’ four-stage process as a basis for discussion of problem solving. Utilizing the psychodynamic lens, the consideration of mathematical creativity is more in the process of problem solving (e.g., Guilford, 1967; Pelczer & Rodriguez, 2011) and less in the product created by said process (e.g., Runco & Jaeger, 2012). A review of some of the mathematics education literature, organized into Wallas’ four stages, is discussed at the beginning of this article. An emphasis is added on the proving process, seen as a subset of problem solving (Furinghetti & Morselli, 2009; Weber, 2005), since most mathematicians, graduate students, and upper-level undergraduates students employ problem solving in their proving. A discussion of teaching observations incorporating the four stages in the undergraduate mathematics classroom, with possible future research including projects using new technology, conclude the article.

Wallas' Four-stage Creative Process

The Wallas model is categorized as a *psychodynamic* approach in Sternberg's (2000) six approaches of creativity. According to Freiman and Sriraman (2007), "the psychodynamic approach to studying creativity is based on the idea that creativity arises from the tension between conscious reality and unconscious drives" (p. 24). There is a dynamic interplay between consciousness and subconscious/nonconscious, hence the term "psychodynamic."

Wallas' psychodynamic creative process has been previously verified by researchers. By interviewing mathematicians, Sriraman (2004) found that a similar four-stage creative process often occurred. However, it seems as though there are non-anecdotal difficulties with measuring the psychodynamic approach (Liljedahl, 2004), partly due to the difficulty of capturing subconscious or nonconscious. The 21st century may bring technological innovations to research methodology (e.g., neuroscience methods, tablet/Livescribe™ pens) to further probe this approach.

Other researchers have demonstrated that impasses are important in psychodynamic mathematical creativity, since impasses might generate a break in problem-solving and allow the subconscious to play. Mason, Burton, and Stacey (1982) suggest, "there is nothing wrong with being unable to make progress on a question, and there is a tremendous value in tussling with it, rephrasing it, distilling it, mulling it over, and modifying it in various ways" (p. 142). Ervynck (1991) stated that:

> What is essential in the individual is a state of mind prepared for mental activity that relates previously unrelated concepts. [Mathematical creativity] is often observed to occur after a period of intense activity involving a heightened state of consciousness of the context and all the constituent parts. And yet it is more likely to bear fruit when the mind is subsequently relaxed and able, subconsciously, to relate the ideas in a manner which benefits from quiet, unforced, contemplation. (p. 44)

Impasses seem to be an important aspect of mathematics; if any mathematician did not experience an impasse, then Fermat's Last Theorem may have been proved years ago. But one must have "intense activity" in order for an impasse, and subsequent subconscious work, to occur. What problem-solving aspects do that "intense activity", or the preparation stage, encompass?

Preparation

The *Preparation* stage of Wallas' model "focuses the individual's mind on the problem and explores the problem's dimensions," (Baker & Czarnocha, 2015, p. 4) Haylock (1987) described preparation as the stage where "the problem is investigated thoroughly and consciously, and familiarity with all its aspects is obtained" (p. 63). This may be the most important of the four stages: without it, no problem solving occurs, and with little preparation, there may be no way for the brain to take advantage of the other three stages. Poincaré (1958) believed that the "preparation stage" along with incorrect proving attempts on proofs is more useful than one usually thinks, believing it sets the unconscious mind at work.

Preparation can be observed to exist in many different forms in different frameworks. Developing a framework influenced by the work of Pólya (1957), Carlson and Bloom (2005) stated that their first stage in problem solving is "orientation," which they defined as "initially engag[ing] in intense efforts to make sense of the information in the problem" (p. 68). Both "understanding a problem" and "making sense of the information" are important in the preparation stage, but planning, executing, and checking (the other three stages in Carlson and Bloom's (2005) framework) may also be involved in the preparation stage defined by Wallas. For example, after orientation, one may immediately attempt (plan and execute) a solution to a problem, verify that the solution is incorrect, and be back at the planning stage, thus exploring more of the problem. In fact, if an individual perceives that a solution is correct after the first attempt with relative ease, this problem may not have been a problem in the sense of Schoenfeld (1985), but rather an exercise for the solver. Hence, the preparation stage may discern whether a problem was an exercise or a true "problem" for a person.

However, there are some instances in which mathematicians attempt a problem for hours. This requires many different iterations of the planning-executing-checking cycle, which may convert a problem into an exercise without the need for a "break." Silveira (1972) found in an empirical psychological study that "problem solvers performed better with longer preparation *and incubation periods* [emphasis added]" (Sio & Ormerod, 2009, p. 96). When one has exhausted the exploration of the problem's dimensions, one may tend to take a (perhaps a well-deserved) break.

Incubation

That break, according to Wallas' four stages, is called *incubation*. Incubation is defined as when "the problem is internalized into the unconscious mind and nothing appears externally to be happening" (Baker & Czarnocha, 2015, p. 4). Incubation has also been described as "a gradual and continuous unconscious process . . . during a break in the attentive activity toward a problem" (Segal, 2004, p. 141).

What happens in the mind during the incubation period? Neuroscientists have been investigating what the brain does physically (blood oxygenation levels via fMRI) or electrically (neural impulses via EEG) during the incubation period. For example, there have been different fMRI studies on incubation in problem solving (e.g., Binder, et al., 1999; De Luca, Beckmann, De Stefano, Matthews, & Smith, 2006). Researchers found that during incubation "the brain contains highly organized, spontaneous patterns of functional activity at rest" (Buckner & Vincent, 2007, p. 2), which provides evidence for incubation being a somewhat successful problem-solving action.

Incubation has its roots in the psychology literature. For example, Segal (2004) conducted an experiment to investigate whether breaks (incubation periods) from problem-solving could be helpful, and what difficulty of cognitive tasks might be introduced during an incubation period. Segal concluded that "a break would improve the performance of insight problem-solving [the psychodynamic approach], but that the duration of the break would not influence performance" (p. 147). On cognitive tasks, Segal stated, "less demanding activity during the break serves as a weaker diversion" more than higher-demanding tasks. The hypothesis about higher cognitive-demand tasks during the incubation period has been supported in other studies (e.g., Kaplan, 1990; Patrick, 1986). However, many of the tasks that have been used to generate incubation (and subsequent insight) involved laboratory problems of time lengths from one to 60 minutes (Sio & Ormerod, 2009). In contrast, many mathematicians, and perhaps even undergraduate or graduate students, incubate in proving or problem-solving for more than 60 minutes, and perhaps also in other settings (home, park, theater, etc., (Savic, 2015)). Therefore, while the field of psychology has much to offer in terms of vocabulary and motivation, experiments investigating mathematical creativity may need to be modified in order to maximally capture problem solving in a more natural setting.

Savic (2015) attempted to address the issue of exploring mathematicians' problem-solving process in a more natural environment by equipping nine mathematicians with Livescribe™ pens, capable of capturing synchronized writing and audio and providing them with challenging mathematical tasks. The resulting data was uploaded to a computer, and date and time stamps were associated with each synchronized proving session. Six of the nine mathematicians experienced some impasses in their proving, and all six engaged in an incubation period of various lengths. Exit interviews with the mathematicians exposed that some incubation periods yielded illuminations or AHA! moments, and that mathematicians had established procedures for where and how they would engage in incubation. For example, one mathematician, Dr. G, talked about taking a walk, and stated that this was common when he encountered an impasse. He stated:

> When [mathematicians] are working on something, we are usually scribbling down on paper. When you go take a break, . . . you are thinking about it in your head without any visual aids . . . [walking around] forces me to think about it from a different point of view, and try different ways of thinking about it, often [from] global, structural points of view. (Savic, 2015, p. 75)

The incubation stage tends to be the most difficult to acknowledge and investigate. Many studies have investigated incubation via self-reported retrospective evidence by interviewing mathematicians (Sriraman, 2004) or students (Garii, 2002). Another difficulty of investigating incubation is that it is usually coupled with illumination or insight. When a problem solver has a sudden insight or AHA! moment, it is usually after a period of incubation. Hence, acknowledgment of a period of incubation must occur *after* an insight. Aiken (1973, p. 409) added a caveat of "success" to the definition of incubation and insight by stating, "if the prolonged unconscious work of the second stage is successful, a third stage occurs-illumination or insight into a solution." I respectfully disagree with Aiken on one aspect; insight does not have to be "correct" for the acknowledgment of incubation. However, a question may be posed: What defines the "end" of incubation?

Illumination

The idea that arises from an incubation period is called an *illumination*. Sometimes called insight, illumination is defined by Baker and Czarnocha (2015) as "where the creative idea bursts forth from its preconscious processing into conscious awareness" (p. 5). Leikin (2014) stated that "insight exists when a person acts adequately in a new situation, and as such, insight is closely related to creative ability" (p. 249). Liljedahl (2013) stated that "illumination is THE aspect of the process that sets creativity, discovery, and invention apart from the more ordinary, and more common, processes of solving a problem—it is the marker that something remarkable has taken place" (p. 255). Therefore, one may associate many "product" definitions of mathematical creativity (e.g., "originality or effectiveness" (Runco & Jaeger, 2012, p. 92)) with insights, since the insight is the (unverified) idea of a solution for a problem or theorem, and hence a mathematical product.

Mathematicians have described *insight* as "seeing a connection," "the light switches," and "having a feel for how things connect together" (Burton, 1999b, p. 28). How does one obtain this illumination? Many believe it is the subconscious "at play" when the person is in a state of incubation. It is sometimes referred to as the "AHA Moment" (Liljedahl, 2004), due to its unexpected nature, coupled with a sense of euphoria or positive emotion (Burton, 1999a). Liljedahl (2013, p. 264) stated that the affective component of illumination is what sets it "apart from other mathematical experiences." Also, the person must have some conviction (Poincaré, 1946) that the illumination has value towards the solution or proof, or else there would be no elation from an AHA moment.

The moment of insight has been described as an aspect of Koestler's (1964) bisociation theory: "the spontaneous leap of insight...which connects previously unconnected matrices of experience [frames of reference] and makes us experience reality on several planes at ones" (p. 45; cited in Baker & Czarnocha, 2015). Sriraman (2004, p. 30) conjectured that "the mind throws out fragments (ideas) which are products of past experience." Somehow the mind, when problem solving, subconsciously keeps piecing together information, perhaps even digging into long-term memory to create a solution or proof. However, that AHA moment for a solution or proof may be invalid.

Verification

The *verification* stage is needed to acknowledge that the illumination is valid. Verification is defined by Baker and Czarnocha (2015) as "where the idea is consciously verified, elaborated, and then applied" (p. 5). It is the stage in which an illumination is confirmed or refuted, including many of the little details that may not have been fully checked in one's mind. In the Carlson and Bloom (2005) multidimensional problem-solving framework, the two phases "executing" and "checking" are both located in the verification stage, since the executing stage is when writing occurs in order to expand on ideas created in the planning stage, and the checking stage is when one examines for errors what s/he has written.

Liljedahl (2004) cautioned that the verification stage is not only about validity of the solution: “it is also a method by which the solver re-engages with the problem at the level of details” (p. 16). In fact, one could view verification as metacognitive, where one may also be looking for uses of the problem-solving or proving technique for other problems, or posing questions that may develop from the solution. Knuth (2002) stated that “much is to be gained by making the solution or an aspect of the problem a starting point for mathematical exploration – exploration that lies at the heart of mathematical practice” (p. 130).

In the proof literature, verification may be discussed as proof validation, defined as “the process an individual carries out to determine whether a proof is correct and actually proves the particular theorem it claims to prove” (Selden & Selden, 1995, p. 127). Proof validation is a subset of proof comprehension, which considers “understanding the content of the proof and learning from it” (Mejia-Ramos & Weber, 2011, p. 331). To validate, one must understand the content of the proof. While proof validation has been examined in the literature (e.g., Selden & Selden, 2003), proof self-validation, which I consider the verification stage of the proving process, has rarely been examined in detail.

Verification is the stage that may differentiate creativity in mathematics from other disciplines. According to Sadler-Smith (2015), Wallas had a three-stage creativity process until he used the writings of Poincaré to add the fourth stage, stating that “incubation supplied a starting point for further work in the verification stage” (p. 344). Poincaré himself stated that “It usually happens that it [the illumination] does not deceive him [the mathematician], but it also sometimes happens, as I have said, that it [the illumination] does not stand the test of verification” (Sadler-Smith, 2015, p. 344). If one does not have a success in the verification stage, the cycle comes back to either preparation (“What am I missing?”) or incubation (“Maybe I need another break?”).

Mathematical Creativity Teaching Observations

Wallas’ four stages may be happening in undergraduate mathematics; however, there seems to be little evidence for its explicit discussion in the classroom. Below, I outline three observations that I believe may be occurring in university courses that may assist with discussing mathematical creativity in the classroom. These observations come from experience teaching undergraduate classes and the shortage of research literature about mathematical creativity and Wallas’ four stages at the undergraduate level.

Observation 1: A majority of tasks posed may not allow for the four stages to occur.

Schoenfeld (1989) surveyed 206 high school students on the length of time it took to solve a typical homework problem, and the average was just under 2 minutes, and not one of the 206 students “allotted more than 5 minutes” (p. 345). Coincidentally, in Lithner’s (2004) calculus textbook study, he claimed that over 70% of the problems can be solved using examples in the text. According to Selden and Selden (2013), in proof-based courses, several tasks are “Type 1,” where the “proofs…can depend on a previous result in the notes” (p. 320), as opposed to “Type 2,” defined as “require formulating and proving a lemma not in the notes, but one that is relatively easy to notice, formulate, and prove” (p. 320) and “Type 3,” which is hard to notice the lemma needed. This may convince a student that one needs to arrive at the solution or proof immediately.

Tasks of this type may not reflect how mathematics is generally practiced (Burton, 1999b). However, there are articles that have discussed tasks that allow for the potential of creativity (Leikin, 2014). Zazkis and Holton (2009) posed tasks from topics such as graph theory and number theory. One of their tasks is stated: "Prove that $n^5 - n$ is divisible by 3" (p. 348). Silver (1997) stated that perhaps students might need "complex, ill-structured problems" (p. 77) in order to engage in the mathematical creative process. Regardless of the topic of task, students may need some tasks that allow them to incubate and illuminate, since they may have situations later in life that may push them to participate in the creative process (Tomasco, 2010). The difficulty with assigning these tasks is that instructors then open up the classroom for a variety of problem-solving strategies, some of which the instructor may not be prepared for, or may be completely incorrect. Therefore, the next observation may help alleviate this difficulty.

Observation 2: Many students do not have classroom environments where impasses and productive failure is encouraged by their instructors.

While tasks play a part in fostering an environment, the instructor's actions, both in the classroom and in feedback, play another crucial role. Actions in the classroom can include think-pair-share periods, defined as "a multi-mode discussion cycle in which students have time to *think* individually, talk with each other in *pairs*, and finally *share* responses with the larger group" (McTighe & Lyman, 1988, p. 19). Think-pair-share may encourage students that are shy with the large group to see that others are experiencing impasses, and hopefully exchange ideas for overcoming such impasses.

When presented with a non-traditional solution from a student, an instructor has many choices to make, including how to address the solution presented, what to do with some students who did not understand the solution, and so forth. However, the instructor may have pre-conceived solutions that s/he wanted to emphasize. This scenario seems all too common in the mathematics classroom, but how the scenario is settled has ramifications for how the social and sociomathematical norms (Yackel & Cobb, 1996) are fostered. Hershkovitz, Pelled, and Littler (2009) stated that "a teacher who has some pre-determined answer expectations might easily suppress creative initiatives by not accepting or ignoring children's ideas" (p. 265). This suppression may have long-lasting effects: Students that might have shared their ideas may be "scared" or "afraid," thus silencing a majority of students that may have creative solutions to certain problems. Mistakes and productive failure can be incredibly important in education (Burger & Starbird, 2012), and can be used in the classroom as "springboards to inquiry," allowing students to "go beyond diagnosis and remediation" (Borasi, 1994, p. 167). Instead, to encourage mathematical creativity, instructors "should encourage good ideas even (and, in fact, especially) when a student suggests an unexpected answer or when the answers are inaccurate" (Hershkovitz, Peled, & Littler, 2009, p. 265).

Observation 3: Many students have not developed actions to cope with impasses, that is, students may not have found their "incubation" or "preparation" actions.

The author (Savic, 2012; Savic, 2015) studied both mathematicians and graduate students proving theorems, focusing on their incubation periods. A majority of the mathematicians had certain activities that they engaged in specifically for incubation, including watching TV, sleeping, or going on a walk in a certain park. In contrast, none of the five graduate students had an activity to implement when incubating. When interviewed, and pressed about incubation, they spoke of taking breaks in a general manner. In combination with Schoenfeld's (1989) study about high school students and time spent on homework, perhaps many undergraduate/graduate students may not have experienced many situations where incubation needed to take place. Therefore, encouraging students to have activities or places of incubation may help their problem-solving processes.

The "preparation" stage may be lacking in some students. In proving, preparation rarely results in the key idea of a proof; often "unpacking" (in the sense of Selden and Selden, 1995) definitions may yield a valid proof of a Type 1 theorem (Selden and Selden, 2013). However, some students may not engage in the proving process at all by avoiding certain theorems, or if they do, their lack of persistence may be a block for successful proving. Our research group created a rubric, named the Creativity-in-progress Rubric (CPR) on Proving (Savic, Karakok, Tang, El Turkey, & Naccarato, in press; Karakok, Savic, Tang, & El Turkey, in press), to consider actions in the preparation stage of the proving process that may yield the potential to be creative.

There are two main categories of the CPR on Proving: *making connections* and *taking risks*. *Making connections* is defined as an ability to connect the proving task with definitions, theorems, multiple representations, and examples from the current course that a student is in, and possible prior experiences from previous courses. The sub-categories involve making connections: between definitions/theorems, between representations, and between examples. *Taking risks* is defined as an ability to actively attempt a proof, demonstrate flexibility in using multiple approaches or techniques, posing questions about reasoning within the attempts, and evaluating those attempts. The sub-categories involve: tools and tricks, flexibility, posing questions, and evaluation of the proof attempt. Every sub-category has three different levels: beginning, developing, and advancing. Each level has a small description to help either the instructor or student know what level s/he is in, and what s/he can achieve (See Appendix A for the full rubric and details).

The actions outlined in the CPR on Proving allow students to explicitly examine their own proving process and adjust accordingly. Self-evaluation of the proving process may help create persistence, and hence situate the student in the preparation stage, with a better chance for incubation (conscious and sub-conscious thoughts) to have more effect. Incubation cannot be only recommended, taught, or self-forced. For students to allow incubation (and subsequently insight) to occur, there must be a certain amount of preparation. This preparation cannot be measured by time, but perhaps by perseverance on the solution (almost to exhaustion) coupled with a curiosity. I contend that mathematical curiosity (Knuth, 2002) may be, ultimately, most important for a student, both in the short-term with finding a solution, and in the long-term with his/her penchant to attack more problems with less difficulty or resistance.

Conclusions/Future Research

In conclusion, Wallas' (1926) four stages of the creative process (preparation, incubation, illumination, and verification), brought into the mathematical world by Hadamard (1945), seem to permeate the literature of both psychology and mathematics education. Preparation is the most important stage of the four: Without preparation, there is no engagement in the other three stages. Investing many of one's mental resources may not lead one to a satisfactory solution to a problem; therefore, one may recognize this as an impasse and take a "break." Incubation allows conscious and subconscious ideas to merge, signaling the brain when a spontaneous solution has been brought to consciousness. This AHA! moment or insight is the illumination stage, where a creative idea is found and used. The validation stage identifies if the insight is correct; if not, either the person cycles back to the preparation stage, or goes into another incubation stage.

Teaching the psychodynamic approach of mathematical creativity may be difficult. An instructor must consider merging three different dimensions of the classroom in harmony for a student to participate in the four stages:

1) Some tasks must allow incubation to occur.
2) Pedagogical moves may need to permit impasses and productive failure to occur in the classroom.
3) Instructors could assist students in finding personal actions that foster the preparation and incubation stages.

There are three future research directions of the creativity in undergraduate mathematics research group that created the CPR on Proving. First, investigation in the implementation of the CPR on Proving in the classroom is a priority. How are instructors and students utilizing the rubric, and what teaching materials can accompany the rubric? Secondly, the group would like to explore the social facet of creativity in proving with both mathematicians and undergraduate students. Is there social activity that fosters creativity in collaboration? Finally, focusing on Wallas' four stages, the author and another colleague are collaborating with the neuroscience division at our university to investigate brain activity in the proving process, including what insights may occur, whether insight is correlated with key ideas of a proof (in the sense of Raman, 2003), and whether one can predict, along with other factors, when a student may have an insight in his/her proving process. With the invention of neuroeducational theory (Anderson, 2014), coupled with new naturalistic data collection techniques such as LiveScribe™ pens (Savic, 2015), the future is bright for research in mathematical creativity *and* problem solving.

References

Aiken, L. (1973). Ability and creativity in mathematics. *Review of Educational Research*, 405-432.

Anderson, O. R. (2014). Progress in application of the neurosciences to an understanding of human learning: The challenge of finding a middle-ground neuroeducational theory. *International Journal of Science and Mathematics Education, 12*, 475-492.

Baker, W., & Czarnocha, B. (2015). AHA! moment, bisociation and simultaneity of attention. *Congress of European Research in Mathematics Education – 9 Proceedings.* Prague.

Binder, J. R., Frost, J. A., Hammeke, T. A., Bellgowan, P. S., Rao, S. M., & Cox, R. W. (1999). Conceptual processing during the conscious resting state: A functional MRI study. *Journal of Cognitive Neuroscience*, 80-93.

Borasi, R. (1994). Capitalizing on errors as" springboards for inquiry": A teaching experiment. *Journal for Research in Mathematics Education*, 166-208.

Burger, E. B., & Starbird, M. (2012). *The 5 elements of effective thinking.* Princeton, NJ: Princeton University Press.

Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37*, 121-143.

Burton, L. (1999). Why is intuition so important to mathematicians but missing from mathematics education? *For the Learning of Mathematics*, *19*(3), 27-32.

Byers, W. (2007). *How mathematicians think: Using ambiguity, contradiction, and paradox to create mathematics*. Princeton, NJ: Princeton University Press.

Carlson, M., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent problem-solving framework. *Educational Studies in Mathematics, 58*, 45-75.

Chamberlin, S. A. (2008). What is problem solving in the mathematics classroom? *Philosophy of Mathematics Education, 23*. Retrieved June 8, 2012, from http://people.exeter.ac.uk/PErnest/pome23/Chamberlin%20What%20is%20Math%20Prob%20Solving.doc.

Craft, A. (2001). *An analysis of research and literature on creativity in education.* Report prepared for the Qualifications and Curriculum Authority.

De Luca, M., Beckmann, C. F., De Stefano, N., Matthews, P. M., & Smith, S. M. (2006). fMRI resting state networks define distinct modes of long-distance. *NeuroImage, 29*, 1359-1367.

Ervynck, G. (1991). Mathematical creativity. In D. Tall, *Advanced Mathematical Thinking* (pp. 42-52). New York, NY, USA: Kluwer Academic Publishers.

Freiman, V., & Sriraman, B. (2007). Does mathematics gifted education need a working philosophy of creativity? *Mediterranean Journal for Research in Mathematics Education, 6*, 23-46.

Furinghetti, F., & Morselli, F. (2009). Every unsuccessful problem solver is unsuccessful in his or her own way: Affective and cognitive factors in proving. *Educational Studies of Mathematics, 70*, 71-90.

Garii, B. (2002). *That "Aha" experience: Meta-cognition and student understanding of learning and knowledge.* Retrieved May 15, 2012, from http://eric.ed.gov/PDFS/ED464127.pdf

Guilford, J. P. (1967). *The nature of human intelligence.* New York, NY, U.S.A.: McGraw-Hill.

Hadamard, J. (1945). *The mathematician's mind.* Princeton: Princeton University Press.

Haylock, D. (1987). A framework for assessing mathematical creativity in school children. *Educational Studies in Mathematics*, *18*(1), 59-74.

Hershkovitz, S., Peled, I., & Littler, G. (2009). Mathematical creativity and giftedness in elementary school: Task and teacher promoting creativity for all. In R. Leikin, A. Berman, & B. Koichu, *Creativity in Mathematics and the Education of Gifted Students* (pp. 255-269). Sense Publishers.

Kaplan, C. (1990). *Hatching a theory of incubation: Does putting a problem aside really help? If so, why?* Carnegie Mellon University.

Karakok, G., Savic, M., Tang, G., & El Turkey, H. (in press). Mathematicians' views on undergraduate student creativity. *Congress of European Research in Mathematics Education – 9 Proceedings.* Prague.

Knuth, E. (2002). Fostering mathematical curiosity. *The Mathematics Teacher, 95*(2), 126-130.

Koestler, A. (1964). *The Act of Creation.* London: Hutchinson & Co (publishers), LTD.

Leikin, R. (2014). Challenging mathematics with multiple solution tasks and mathematical investigations in geometry. In Y. Li, E. A. Silver, & S. Li, *Transforming mathematics instruction: Multiple approaches and practices* (pp. 59-80). Dordrecht, the Netherlands: Springer.

Leikin, R. (2014). Giftedness and high ability in mathematics. In S. Lerman, *Encyclopedia of mathematics education* (pp. 247-251). Springer Netherlands.

Liljedahl, P. (2004). *The AHA! experience: Mathematical contents, pedagogical implications.* (Doctoral Dissertation). Vancouver: Simon Frasier University.

Liljedahl, P. (2009). In the Words of the Creators. In R. Leikin, A. Berman, & B. Koichu, *Mathematical Creativity and the Education of Gifted Children* (pp. 51-70). Rotterdam: Sense Publishers.

Liljedahl, P. (2013). Illumination: an affective experience? *ZDM, 45*(2), 253-265.

Lithner, J. (2004). Mathematical reasoning in calculus textbook exercises. *The Journal of Mathematical Behavior, 23,* 405-427.

Mann, E. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted, 30*(2), 236-260.

McTighe, J., & Lyman, F. T. (1988). Cueing thinking in the classroom: The promise of theory-embedded tools. *Educational Leadership, 45*(7), 18-24.

Mejia-Ramos, J. P., & Weber, K. (2011). Why and how mathematicians read proofs: An exploratory study. *Educational Studies in Mathematics, 76*(3), 329-344.

Patrick, A. S. (1986). The role of ability in creative "incubation". *Personality and Individual Differences, 7*(2), 169-174.

Pelczer, I., & Rodriguez, F. G. (2011). Creativity assessment in school settings through problem posing tasks. *The Montana Mathematics Enthusiast, 8,* 383-398.

Poincaré, H. (1946). *The foundations of science.* Lancaster, PA: The Science Press.

Polya, G. (1957). *How to solve it: A new aspect of mathematical method.* Garden City, NJ: Doubleday.

Prusak, A. (2015). Nurturing students' creativity through telling mathematical stories. *Proceedings of the 9th International MCG Conference,* (pp. 16-21). Sinaia, Romania.

Raman, M. (2003). Key Ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics, 52,* 319-325.

Runco, M. A., & Jaeger, G. G. (2012). The standard definition of creativity. *Creativity Research Journal*, *24*(1), 92-96.

Sadler-Smith, E. (2015). Wallas' Four-Stage Model of the Creative Process: More Than Meets the Eye? *Creativity Research Journal*, *27*(4), 342-352.

Savic, M. (2012). *Proof and Proving: Logic, Impasses, and the Relationship to Problem Solving*. (Doctoral Dissertation): New Mexico State University.

Savic, M. (2015). The incubation effect: How mathematicians recover from proving impasses. *Journal of Mathematical Behavior*, *39*, 67-78.

Savic, M., Karakok, G., Tang, G., El Turkey, H., & Naccaratto, E. (in press). Formative Assessment of Creativity in Undergraduate Mathematics: Using a Creativity-in-Progress Rubric (CPR) on Proving. In R. Leikin, & B. Sriraman, *Creativity and Giftedness: Interdisciplinary Perspectives from Mathematics and Beyond*. Springer.

Schoenfeld, A. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for Research in Mathematics Education*, *20*(4), 338-355.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334-370). New York, NY: Macmillian.

Schumacher, C. S., & Siegel, M. J. (2015). *2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences*. Washington, DC: Mathematical Association of America.

Segal, E. (2004). Incubation in insight problem solving. *Creativity Research Journal*, *16*, 141-148.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education, 34,* 4-36.

Selden, A., & Selden, J. (2013). Proof and problem solving at the university level. *The Mathematics Enthusiast, 10*(1&2), 303-334.

Selden, J., & Selden, A. (1995). Unpacking the logic of mathematical statements. *Educational Studies in Mathematics, 29*, 123-151.

Silveira, J. (1972). *Incubation: The effect of interruption timing and length on problem solution and quality of problem processing.* Unpublished doctoral dissertation, University of Oregon.

Silver, E. (1997). Fostering creativity through instruction rich in mathematical problem solving and posing. *ZDM Mathematical Education, 3*, 75-80.

Sio, U. N., & Ormerod, T. C. (2009). Does incubation enhance problem solving? A meta-analytic review. *Psychological Bulletin, 35*, 94-120.

Sriraman, B. (2004). The characteristics of mathematical creativity. *The Mathematics Educator, 14*, 19-34.

Sternberg, R. J. (2000). *Handbook of Creativity*. Cambridge University Press.

Tomasco, S. (2010, May 18). *IBM 2010 Global CEO Study: Creativity Selected as Most Crucial Factor for Future Success*. Retrieved March 19, 2016, from IBM: http://www-03.ibm.com/press/us/en/pressrelease/31670.wss

Wallas, G. (1926). *The art of thought.* New York: Harcourt Brace.

Weber, K. (2005). Problem-solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in the activity of proof construction. *Journal of Mathematical Behavior, 24,* 351-360.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 458-477.

Zazkis, R., & Holton, D. (2009). Snapshots of creativity in undergraduate mathematics education. In R. Leikin, A. Berman, & B. Koichu, *Creativity in mathematics and the education of gifted students* (pp. 345-365). Rotterdam, the Netherlands: Sense.