Where is the Logic in Student-Constructed Proofs?

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Often university mathematics departments teach some formal logic early in a transition-to-proof course in preparation for teaching undergraduate students to construct proofs. Logic, in some form, does seem to play a crucial role in constructing proofs. Yet, this study of forty-two student-constructed proofs of theorems about sets, functions, real analysis, abstract algebra, and topology, found that only a very small part of those proofs involved logic beyond common sense reasoning. Where is the logic? How much of it is just common sense? Does proving involve forms of deductive reasoning that are logic-like, but are not immediately derivable from predicate or propositional calculus? Also, can the needed logic be taught in context while teaching proof-construction instead of first teaching it in an abstract, disembodied way? Through a theoretical framework emerging from a chunk-by-chunk analysis of student-constructed proofs and from task-based interviews with students, I try to shed light on these questions.

Keywords: Proofs, logic, transition-to-proof courses, analysis of proofs, task-based interviews

Why is this an interesting question? To obtain a Masters or Ph.D. in mathematics, or even to succeed in proof-based courses in an undergraduate mathematics major, one must often be able to construct original proofs, a common difficulty for students (Moore, 1994; Weber, 2001). This process of proof construction is usually explicitly taught, if at all, to undergraduates as a small part of a course, such as linear algebra, whose stated goal is something else, or in a transition-to-proof or "bridge" course. When universities do offer a transition-to-proof course, professors often teach some formal logic (predicate and propositional calculus) as a background for proving. But how much logic actually occurs in student-constructed proofs? In this paper, I begin to answer this question by first searching for uses of logic in a "chunk-by-chunk" analysis of student-constructed proofs from a graduate "proofs course," then by coding student-constructed proofs from a graduate homological algebra course, and lastly by examining the actions in the proving process of three graduate students and searching for additional uses of logic therein. If formal logic occurs a substantial amount, then teaching a unit on predicate and propositional calculus might be a good idea; however, if formal logic occurs infrequently, then teaching it in context, while teaching proving, may be more effective.

Background Literature

Currently, at the beginning of transition-to-proof courses, professors often include some formal logic, but how it should be taught is not so clear. Epp (2003) stated that, "I believe in presenting logic in a manner that continually links it to language and to both real world and mathematical subject matter" (p. 895). However, some mathematics education researchers maintain that there is a danger in relating logic too closely to the real world: "The example of 'mother and sweets' episode, for instance, which is 'logically wrong' but, on the other hand, compatible with norms of argumentation in everyday discourse, expresses the sizeable discrepancy between formal thinking and natural thinking" (Ayalon & Even, 2008b). In the mother and sweets scenario, the mother says to the child, "If you don't eat, you won't get any sweets" and the child responds by saying, "I ate, so I deserve some sweets." Other authors have noticed that the way logic is taught in transition-to-proof courses is at variance with how it is actually used in proving: "Beginning logic courses often seem to present logic very abstractly, in essence as a form of algebra, with examples becoming a kind of applied mathematics" (Selden & Selden, 1999, p. 8).

There are also those who think that logic does not need to be explicitly introduced at all. For example, Hanna and de Villiers (2008) stated, "It remains unclear what benefit comes from teaching formal logic to students or to prospective teachers, particularly because mathematicians have readily admitted that they seldom use formal logic in their research" (p. 311). Selden and Selden (2009) claimed that "logic does not occur within proofs as often as one might expect … [but] [w]here logic does occur within proofs, it plays an important role" (p. 347). Taken together, these differing views suggest that it would be useful for mathematics education researchers to further examine the role of logic and logic-like reasoning within proofs in order to inform professors on the ways they might best include logic in transition-to-proof courses. However, to date, only a little such research has been conducted (Baker, 2001).

Another interesting idea that has been expressed about proofs in general is that deduction occurs in proofs in a "systematic, step-by-step manner" (Ayalon & Even, 2008a). In fact, one professor quoted by Ayalon and Even (2008a) expressed the view that a student "thinks about something, he draws a conclusion, which brings him to the next thing…Logic is the procedural, algorithmic structure of things." Others tend to agree: "From most mathematical textbooks we can simply see the process of a mathematical proof [sic] as the development of a sequence of statements using only definitions and preceding results, such as deductions, axioms, or theorems" (Chin & Tall, 2002, p. 213). Rips (1994) looks at proof in a slightly more sophisticated way: "At the most general level, a formal proof is a finite sequence of sentences $(s_1, \ldots, s_k)$ in which each sentence is either a premise, an axiom of the logical system, or a sentence that follows from the preceding sentences by one of the system's rules" (p. 34). Instead of sentences, I partition student-constructed proofs into usually smaller "chunks" to begin to answer the title question on logic.

Research Settings

This research was done in three separate phases: (a) By examining all of the 42 student-constructed proofs from a beginning graduate level "proofs course," (b) by examining 10 student-constructed proofs from a more advanced graduate homological algebra course, and (c) by conducting and analyzing task-based interviews with three graduate students a year after the "proofs course." The "proofs course," *Understanding and Constructing Proofs*, was offered at a large Southwestern state university, giving Masters and Ph.D.'s in mathematics. Students in the course were first-year mathematics graduate students along with a few advanced undergraduate mathematics majors. For this course, the students were given professor-created notes with a sequence of definitions, questions, and statements of theorems dealing with topics such as sets, functions, real analysis, algebra, and topology. For example, three theorems that were proved by the students were: "The product of two continuous [real] functions is continuous"; "Every semigroup has at most one minimal ideal"; and "Every compact, Hausdorff topological space is regular." The topics in the course were of less importance than its focus on the construction of differing kinds of proofs.

Students were asked to prove theorems in the notes at home, and came to the class to present their proofs on the chalkboard. After receiving critiques on content and style, one student was selected to modify his/her proof to turn in. The professors then verified all of these proofs as correct and photocopies were made for the students. There were no lectures, just discussions of student work. The class met for one hour and fifteen minutes twice a week for a total of 30 class meetings. The course was taught like a modified Moore Method course (Mahavier, 1999) by two professors, using a constructivist approach that was also somewhat Vygotskian because the professors gave their opinions on how mathematicians write proofs. All class meetings were videotaped, and field notes were taken by a graduate research assistant. The debriefing for each class meeting and preparation for the next class were also videotaped, as well as were all tutoring sessions with students.

The homological algebra course was taught lecture-style, with two theorems to prove assigned per week as homework. The professor then graded the proofs and handed them back to the students. The course covered such topics as module theory, category theory, homology and cohomology, with emphasis on the Tor and Ext functors. All students in the course had passed the graduate abstract algebra courses required before taking the Ph.D. algebra comprehensive examination. The ten proofs considered from this course were constructed by a single student who received perfect scores on all homework and actively conducts research in algebra.

The task-based interviews were conducted in a university seminar room one year after the "proofs course." The three graduate students, who participated, had all received A's in the "proofs course" and were willing to volunteer for the one hour interview. Each interviewed student received a one-page subset of the "proofs course" notes. This subset was self-contained and began with the definition of a semigroup and ended with the theorem to be proved, "Every semigroup has at most one minimal ideal." All other theorems listed therein were to be considered as proved and thus could be used by the students in the proving process. Each student was encouraged to think aloud while attempting to construct a proof on the chalkboard. Because of time constraints, after 45 minutes, each student was asked to stop proving and answer some questions about the proving process. The questions were not predetermined. Most were intended to clarify what had been said during the think-aloud process or what had been written on the chalkboard. The interviews were videotaped, and field notes were taken.

Research Methodology

The 42 proofs from the "proofs course" were first subdivided into "chunks" for coding. The "chunks" are similar to those in Miller's (1956) article, in which he stated that chunks are a "meaningful unit" in thinking. In the analysis described here, a chunk can refer to a sentence, a group of words, or even a single word, but always refers to a unit in a proof. During several iterations of the coding process, 13 categories, such as "Informal inference" and "Assumption," emerged. During one iteration, I asked two mathematics professors to observe my coding of two of the student-constructed proofs. After this training, four additional proofs were coded independently by the professors, and over 80% of the chunks were agreed upon by all three of us. It appears that once a person understands this chunking process, he/she can do the chunking with relative ease. This iteration included full agreement on which chunks to associate with specific categories, which helped solidify the coding process.

The categories and the chunks sometimes co-emerged, that is, the categories sometimes influenced the chunking. For example, "Then $x \in A$ and $x \in B$" might have been treated as a single chunk because it arose from $x \in A \cap B$ and the definition of intersection. However, it could have been split into "Then $x \in A$" and "and $x \in B$" because the two chunks seemed to follow from separate warrants. This coding process, which resulted in a total of 673 chunks in the student-constructed proofs from the "proofs course," is further described below.

In this paper, I discuss in detail just 5 of the 13 categories. The first two of these deal with the question posed at the beginning of this paper, "Where is the logic in student-constructed proofs?" The remaining three categories are those that occurred most often. Proofs from both the "proofs course" and the homology course were coded using the same categories. All 13 categories are briefly described in the Appendix.

The Categories

Informal inference (**II**) is the category that refers to a chunk of a proof that depends on common sense reasoning. While I view informal inference as being logic-like, it seems that when one uses common sense, one does so automatically and does not consciously bring to mind any formal logic. For example, given $a \in A$, one can conclude $a \in A \cup B$ by common sense reasoning, without needing to call on formal logic.

By *Formal logic* (**FL**) in this paper I mean the conscious use of predicate or propositional calculus going beyond common sense. The distinction is that formal logic is the logic a student does not normally possess before entering a transition-to-proof course. Modus Tollens and DeMorgan's Laws are two examples of formal logic that are usually not common sense for most students (Anderson, 1980; Austin, 1984). For example, given $x \notin B \cup C$, one can conclude $x \notin B$ and $x \notin C$, a typical use of DeMorgan's Laws that students often do not perform automatically, or do perform automatically, but incorrectly.

Definition of (**DEF**) refers to a chunk in a proof that calls on the definition of a mathematical term. For example, consider the line "Since $x \in A$ or $x \in B$, then $x \in A \cup B$." The conclusion "then $x \in A \cup B$" implicitly calls on the definition of union.

Assumption (**A**) is the code for a chunk that creates a mathematical object or asserts a property of an object in the proof. The category is further divided into two sub-categories: "Choice" and "Hypothesis." *Assumption (Choice)* refers to the introduction of a symbol to represent an object (often fixed, but arbitrary) about which something will be proved – but not the assumption of additional properties given in a hypothesis. In contrast, *Assumption (Hypothesis)* refers to the assumption of the hypothesis of a theorem or argument (often asserting properties of an object in the proof). An example to demonstrate the difference between the two is provided by the theorem "For all $n \in \mathbb{N}$, if $n > 5$ then $n^2 > 25$." The chunk "Let $n \in \mathbb{N}$" would be coded Assumption (Choice), and the chunk "Suppose $n > 5$" would be coded Assumption (Hypothesis).

Interior reference (**IR**) is the category for a chunk in a proof that uses a previous chunk as a warrant for a conclusion. For example, if there were a line indicating $x \in A$ earlier in the proof, then a subsequent line stating "Since $x \in A$…" later in the proof would be an interior reference.

Examples of the Coding

To fully understand the coding and categorizing process, some examples might be helpful:

Theorem 2: For sets A, B, and C, if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.
Proof: Let A, B, and C be sets. Suppose $A \subseteq B$ and $A \subseteq C$. Suppose $x \in A$. Then $x \in B$ and $x \in C$. That means by the def of intersection $x \in B \cap C$. Therefore, $A \subseteq B \cap C$.

Let A, B, and C be sets.	Assumption (Choice)

Suppose $A \subseteq B$ and $A \subseteq C$.	Assumption (Hypothesis)
Suppose $x \in A$.	Assumption (Hypothesis)
Then $x \in B$	Informal inference
and $x \in C$.	Informal inference
That means by the def of intersection $x \in B \cap C$.	Definition of intersection
Therefore, $A \subseteq B \cap C$.	Conclusion statement/Definition of subset

For the above proof, the first chunk was "Let $A, B,$ and C be sets." This was coded "Assumption (Choice)" because the sets were chosen. The next chunk, "Suppose $A \subseteq B$ and $A \subseteq C$" was coded "Assumption (Hypothesis)" since those suppositions were stated in the theorem as the hypotheses. "Suppose $x \in A$" is a chunk that is the hypothesis of a subproof for proving a subset inclusion, hence it was coded "Assumption (Hypothesis)." Both conclusions "Then $x \in B$" and "and $x \in C$" were coded "Informal Inference" because they can both be argued by common sense. The chunk "That means by the def of intersection $x \in B \cap C$" takes the two previous chunks and uses the definition of intersection. Finally, the chunk "Therefore, $A \subseteq B \cap C$" uses the definition of subset and is also the conclusion of the proof. Most coded chunks were counted as one unit each, however this final chunk was assigned a half unit to each category "Conclusion statement" and "Definition of subset."

Theorem 38: If X is a Hausdorff space and $x \in X$, then $\{x\}$ is closed.
Proof: Let X be a Hausdorff space. Let $x \in X$. Note $\{x\} = X - (X - \{x\})$. Suppose $y \in X$ and $y \neq x$. Because X is Hausdorff, there is an open set P_y for which $y \in P_y$. There is also an open set R_y such that $x \in R_y$ and $P_y \cap R_y = \emptyset$. Suppose $P_y \nsubseteq X - \{x\}$, then $x \in P_y$, but $x \in R_y$. Therefore $x \in P_y \cap R_y$, which is a contradiction. Therefore, $P_y \subseteq X - \{x\}$. Thus for every $y \neq x$ there is an open set P_y where $y \in P_y$ and $P_y \subseteq X - \{x\}$. The union of all P_y is equal to $X - \{x\}$, which is thus an open set. Therefore $\{x\}$ is closed, being the complement of an open set.

Let X be a Hausdorff space.	Assumption (Hypothesis)
Let $x \in X$.	Assumption (Hypothesis)
Note $\{x\} = X - (X - \{x\})$.	Formal logic
Suppose $y \in X$ and $y \neq x$.	Assumption (Choice)
Because X is Hausdorff,	Interior reference
there is an open set P_y for which $y \in P_y$. There is also an open set R_y such that $x \in R_y$ and $P_y \cap R_y = \emptyset$.	Definition of Hausdorff
Suppose $P_y \nsubseteq X - \{x\}$,	Assumption (Hypothesis)
then $x \in P_y$,	Formal logic
but $x \in R_y$.	Interior reference
Therefore $x \in P_y \cap R_y$,	Definition of intersection
which is a contradiction.	Contradiction statement
Therefore, $P_y \subseteq X - \{x\}$.	Formal logic
Thus for every $y \neq x$ there is an open set P_y where $y \in P_y$ and $P_y \subseteq X - \{x\}$.	Conclusion statement

The union of all P_y is equal to $X - \{x\}$,	Informal inference
which is thus an open set.	Definition of topology
Therefore $\{x\}$ is closed, being the complement of an open set.	Conclusion statement/Definition of closed

The first chunk in the above proof that I would like to discuss is the third chunk "Note $\{x\} = X - (X - \{x\})$." This was coded as "Formal logic" because this inference would normally not be automatic for most students, as they would have to formulate the negations first, before concluding the set equality. The fifth chunk, "Because X is Hausdorff," restates the first chunk in the proof, so it was coded as "Interior reference." The eleventh chunk, "which is a contradiction," is just a statement to the reader that the prover has arrived at a contradiction; hence the code "Contradiction statement."

The Interviews

The three interviewed graduate students took three different approaches to the proof, including voicing different concept images for several concept definitions (Tall & Vinner, 1981). For example, for the definition of a "minimal ideal of a semigroup," one student considered Venn diagrams when reflecting on the definition, while the other two students stated in a subsequent debriefing that they had not thought of using a diagram. While all three students' proving approaches were different, none of them proved the theorem correctly. This might have been a result of time constraints.

Another result was that the actions that I had previously hypothesized for the proof construction did not match the actual actions of any of the interviewed students. For example, I had hypothesized that the students would write the assumptions as their first line, leave a space, and then write what was to be proved as their last line (as they had been encouraged to do in the earlier "proofs course"). This is a proving technique (Downs & Mamona-Downs, 2005) that is not often taught. Although all three students wrote "Let S be a semigroup" almost immediately at the beginning of their proofs, thereby introducing the letter S, only one student left a space and wrote the conclusion at the end, after some algebraic manipulations. Another student wrote definitions on scratch work before attempting the proof. She then assumed A and B were minimal ideals, and looked up the definition of a minimal ideal. She subsequently claimed (without justification) that either $A = B$ or $A \cap B = \emptyset$. After correctly using a theorem listed in the provided notes, she concluded $A = B$. She then said, "Ok, I'm done." But in fact, the above lack of justification can be seen as a gap in her proof.

The Results

In the chunk-by-chunk analysis of the proofs in the "proofs course," just 6.5% (44 chunks) of the 673 chunks were Informal inference, and just 1.9% (13 chunks) were Formal logic. However, I found that 30% (203 chunks) were Definition of, 25% (166 chunks) were Assumption, and 16% (108 chunks) were Interior reference. Thus, Definition of, Assumption, and Interior reference accounted for nearly 71% of the chunks in the analyzed proofs. These large percentages brought up the question: Was this due to the somewhat unusual nature of the "proofs course" with its wide spread of topics that entailed the introduction of many definitions? Also, how much difference might there be in the amount of logic used if I were to do a chunk-by-chunk analysis of proofs in a graduate homology course that concentrated on a single topic?

In the subsequent chunk-by-chunk analysis of 10 proofs from the homology course, only 10% (17 chunks) of the 170 chunks were Informal inference, and just 0.6% (1 chunk) was Formal logic. Indeed, I found that 21% (36 chunks) were Definition of, 18% (31 chunks) were Assumption, and 18% (30 chunks) were Interior reference, for a total of almost 57%. This lends some support for the hypothesis that the above large percentage (71%) of Definition of, Assumption, and Interior reference was due to the nature of the "proofs course". However, there is not much difference in the percentages of both formal and informal logic used in the two courses (8.4% vs. 10.6%). The table below shows the chunk categories, complete with the rounded percentages. A brief description of all of the categories, their abbreviations (e.g., **A**, **ALG**, etc.), and examples thereof is given in the Appendix.

	# Chunks	A	ALG	C	CONT	D	DEF	ER	FL	II	IR	REL	SIM	SI
"Proofs class"	673	166	23	56	4	32	203	17	13	44	108	3	3	4
% of chunks		24.7	3.3	8.3	0.6	4.7	30.2	2.5	1.9	6.5	16	0.4	0.4	0.6
Homology	170	31	4	14	0	5	36	21	1	17	30	8	1	2
% of chunks		18.2	2.4	8.2	0	2.9	21.2	12.4	0.6	10	17.7	4.7	0.6	1.2

Discussion

At first glance, these results may seem surprising. While the chunk-by-chunk coding is a convenient tool for a surface analysis of a finished written proof, there are underlying structures to, and within, proofs, such as proof by contradiction. I see these as "logic-like structures" that are not often explained in the predicate and propositional calculus discussed in most transition-to-proof courses. For example, if one wishes to prove "For all $x \in A$, $P(x)$," one starts with "Let $x \in A$" and reasons towards "$P(x)$."

"Logic-like structures" are structures that preserve truth value in an argument, yet are not easily expressible in the language of predicate or propositional calculus. Beginning a proof of a theorem whose conclusion is of the form $P\ or\ Q$ by supposing *not* P and arriving at Q is another example of a logic-like structure. Structuring a proof in this way has the effect of using logic. Also, if one wishes to prove that there is a unique x with the property $P(x)$, one normally begins "Suppose $P(a)$ and $P(b)$" and reasons towards the conclusion "$a = b$." This logic-like structure appears in the proof of the theorem, "Every semigroup has at most one minimal ideal," the same theorem used in the interviews. In fact, all three interviewed students immediately assumed a semigroup S and further assumed that S contained two, or n, minimal ideals − both actions which had been discussed in the "proofs course."

The fact that from the "proofs course," 30% of the chunks were definitions and 25% were assumptions suggests that there is a need to teach undergraduates how to introduce mathematical objects into proofs and how to read and use definitions. Indeed, there have been documented instances of students' struggle with definitions (Edwards & Ward, 2004). What this and the results of the chunk-by-chunk coding suggest is that there needs to be a focus on how to unpack and interpret concept definitions in order to create usable concept images (Tall & Vinner, 1981) with components that can be used directly in proofs. For example, by definition, $A \cup B$ is $\{x | x \in A \text{ or } x \in B\}$, but one usage is if one has "$x \in A$ or $x \in B$" then one can conclude "$x \in A \cup B$." Some students do not immediately connect this definition, with its set notation, to this usage. Bills and Tall (1998) introduced a similar notion saying that a definition is *formally* *operable* for a student if that student "is able to use it in creating or (meaningfully) reproducing a formal argument [proof]" (p. 104).

Another implication for teaching that stems from this research is that because formal logic occurs fairly rarely, one might be able to teach it in context as the need arises. Also there is a possibility that doing so might be more effective.

Future Research

It would be interesting to examine whether the kinds of chunks used in proofs varies by mathematical subject area. For example, would topology have a different distribution of categories of chunks than abstract algebra? Indeed, several mathematics professors have suggested that I code chapters of various textbooks to see how much formal logic occurs in them. Also, it may be that the kind of formal logic taught explicitly at the beginning of many transition-to-proof courses is actually psychologically, and practically, disconnected from the process of proving for students. This disconnect might lead to future difficulties in many of the proof-based courses in students' subsequent undergraduate and graduate programs. An additional interesting question that arises from the "proofs course" itself is: How many beginning graduate students need a course specifically devoted to improving their proving skills?

In future research, one might also look for instances of logic-like structures and techniques in student-constructed proofs. Solow (1982) and Velleman (1994) both discuss logic-like structures and techniques for proving, but many other transition-to-proof books touch on this very briefly, if at all. Can one identify a range of logic-like structures that students most often need in constructing proofs? Further, one might investigate the degree of a prover's automated behavioral knowledge of logic-like structures that could help to reduce the burden on his or her working memory. This might free resources to devote to the problem-solving aspects of proofs. That this might be the case was suggested by Selden, McKee and Selden (2010).

Finally, there may be additional logic that does not appear in a final written proof, but that might occur in the actions of the proving process. This would be interesting to investigate. For example, consider the first theorem in the "proofs course" notes, "For sets A and B, if $A \cap B = A$, then $A \cup B = B$." The hypothesized actions for one student-constructed proof of this theorem might be re-stating the hypothesis, "Let A and B be sets, and suppose $A \cap B = A$" at the beginning of the proof, leaving a space, and then writing the conclusion, "Then $A \cup B = B$," at the bottom of the paper or chalkboard. The next action might be to unpack the conclusion and realize that a set equality requires each set to be a subset of the other. This way of showing set equality is a technique that may not be emphasized at the transition-to-proof level, but is very useful, because something similar occurs in many branches of mathematics. In number theory, for example, proofs showing the equality of two numbers sometimes demonstrate that one number divides the other and vice versa.

The next proving action might be to unpack the definition of subset. In this hypothetical proof construction, one might first show that $A \cup B \subseteq B$ by supposing $x \in A \cup B$ and arguing towards $x \in B$. This is subtle because x could be in $A - B$, or in B. Because of this, one might take cases that cover the two possibilities. Using cases produces another logic-like structure, because logically one must exhaust all possibilities. "Suppose $x \in A$" might be the first case, and since one has the hypothesis, "$A = A \cap B$," one can conclude "$x \in B$." This was written into the student-constructed proof as "Let $x \in A$. Since $A = A \cap B$, then $x \in B$." [This can be warranted by the following argument: From $x \in A$ and $A = A \cap B$, one has $x \in A \cap B$. Then $x \in A$ and $x \in B$, so $x \in B$.] Notice that the implicit step "$x \in A \cap B$" was omitted, leaving the reader to supply a warrant. The other case is “Suppose $x \notin A$,” and in this case, since $x \in A \cup B$, by common sense logic one obtains $x \in B$. Since both possibilities have thus been exhausted, one can conclude $x \in B$, and the first subset inclusion has been proved. One would then go on to prove the other subset inclusion.

Sub-arguments, such as the one in brackets above, are sometimes omitted from proofs when readers are assumed to have sufficient background knowledge of the mathematics. The above implicit sub-argument is very simple. Implicit arguments, including those from the graduate homology course proofs, also seemed to be relatively simple. I conjecture that in contrast, implicit arguments in mathematics journals are often much more difficult for a reader to verify.

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Appendix

The following is a list of all 13 categories used in the chunk-by-chunk analysis. For each category, I first give its abbreviation in parentheses (e.g., (**A**)), followed by its designation (e.g., Assumption), a definition, and an example of how it can be used in a chunk-by-chunk analysis.

(**A**) Assumption: We separate assumption into two sub-categories:

- Choice: The choice of a symbol to represent an object (often fixed, but arbitrary) about which something will be proved – but not the assumption of additional properties given in a hypothesis
- Hypothesis: The assumption of the hypothesis of a theorem or argument (often stating properties of an object in the proof).

Example 1: For the theorem "For all $n \in \mathbb{N}$, if $n > 5$ then $n^2 > 25$."
A (Choice): "Let $n \in \mathbb{N}$" (fixed, but arbitrary)
A (Hypothesis): "Suppose $n > 5$"

Example 2: For the theorem "Let π be the ratio of the circumference of a circle to its diameter. Then $\pi > 3$."
A (Choice): "Let π be the ratio of the circumference of a circle to its diameter"
A (Hypothesis): None

(**ALG**) Algebra: Any high school or computational algebra done in the proof.
Example: "... $|x + 4| - 4 \leq |x| + |4| - 4 = |x|$ ..."
ALG: "$|x + 4| - 4 \leq |x| + |4| - 4$"
ALG: "$|x| + |4| - 4 = |x|$"

(**C**) Conclusion statement: A statement that summarizes the conclusion of a theorem or an argument.
Example: "…So $x \in B$. Therefore $A \subseteq B$."
C: "Therefore $A \subseteq B$."

(**CONT**) Contradiction statement: The conclusion of a proof or argument by contradiction.
Example: "…We found $x \in A$, which is a contradiction."
CONT: "which is a contradiction".

(**D**) Delimiter: A word or group of words signifying the beginning or end of a sub-argument. Common delimiters include "now," "next," "firstly," "lastly," "case 1," "in both cases," "part," "(⇒)," "(⊆)," "base case" (in an induction proof), and "by induction" (in an induction proof).
Example: "…In both cases, we conclude that $x \in B$."

D: "In both cases".

(**DEF**) Definition of: The use of a definition of a mathematical object.
Example: "…so $x \in A$… also $x \in B$… Then $x \in A \cap B$…"
DEF: "Then $x \in A \cap B$".

(**ER**) Exterior reference: A reference to a theorem proved previously.
Example: "…Now, by Theorem 6, $x \in A$…"
ER: "by Theorem 6"

(**FL**) Formal logic: Any logic that is not common sense.
Example: "…If $x \in A$ and $x \notin B$, and $A, B \subseteq X$, then $x \notin (X - A) \cup B$…"
FL: "then $x \notin (X - A) \cup B$"

(**II**) Informal inference: An inference depending on common sense logic.
Example: "…If $a \in A$ … $A \subseteq B$… then $a \in B$…"
II: "then $a \in B$"

(**IR**) Interior reference: A chunk of a proof that calls on anything stated earlier in the proof.
Example: "…Let $x \in A$ … $A \subseteq B$… Since $x \in A$, $x \in B$…"
IR: "Since $x \in A$"

(**REL**) Relabeling: Giving an object a new (usually shorter) label.
Example: "…Thus $e_a = e_b$ is the identity. Set $e = e_a = e_b$…"
REL: "Set $e = e_a = e_b$"

(**SI**) Statement of intent: A small statement in a proof that indicates what is intended in the rest of the argument.
Example: "…We want to show that $x \in A$…"
SI: "We want to show that $x \in A$"

(**SIM**) Similarity in Proof: An indication that a section of a proof can be repeated with the same arguments previously given for another part of the proof.
Example: "…Therefore A is a left ideal. Similarly, A is a right ideal…"
SIM: "Similarly, A is a right ideal"