WHAT DO MATHEMATICIANS DO WHEN THEY HAVE A PROVING IMPASSE?

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This paper reports what six mathematicians did when they came to impasses while constructing proofs on an unfamiliar topic, from a set of notes, alone, and with unlimited time. Detailed information is given on two of the mathematicians. By an impasse, I mean a period of time during the proving process when a prover feels or recognizes that his or her argument has not been progressing and that he or she has no new ideas. What matters is not the length of time but its significance to the prover and his or her awareness thereof. I point out two kinds of actions these mathematicians took to recover from their impasses: one kind relates directly to the ongoing argument, while the other kind consists of doing something unrelated, either mathematical or non-mathematical. Data were collected using technology and a new technique being developed to capture individuals' autonomous proof constructions in real-time.

Key words: university level, proof, mathematicians, impasse, data collection technique

This preliminary report presents findings from part of an ongoing larger study of mathematicians, graduate students, and undergraduates constructing proofs on an unfamiliar topic, from a set of notes, alone, and with unlimited time. During separate data collection sessions with nine mathematicians (eight males and one female), six of the nine experienced a considerable period during which they made little progress and developed no new ideas in proving certain theorems. The study investigated what actions these mathematicians took to try to continue, as well as what they later indicated they normally do in such situations. Data were collected employing a new data collection technique being developed to capture individuals' autonomous proof constructions in real-time using a tablet computer or using a LiveScribe pen with special paper. Semi-structured exit interviews were conducted after each mathematician's proving session, followed by focus group reflective interviews conducted with the four professors who used the tablet PC and separately with four of the professors who used a LiveScribe pen (one professor using the LiveScribe pen could not attend the focus group interview). Results, that is, an understanding of how mathematicians recover from periods of no progress and no new ideas, is likely to play a role in facilitating students' learning of proof construction.

Background Literature

Research on mathematicians' practices has been conducted largely in order to guide teaching. Weber (2008) stated that, "investigations into the practices of professional mathematicians should have a strong influence on what is taught in mathematics classrooms" (p. 451). Burton (1999) studied mathematicians specifically for their "attitudes, beliefs and practices…when engaged upon a research problem" (p. 121). Misfeldt (2003) examined how mathematicians make use of writing and re-writing in their own research. Carlson and Bloom (2005) investigated mathematicians' practices during problem solving, an important component of constructing proofs. More recently, Wilkerson-Jerde and Wilensky (2011) explored mathematicians' learning of new mathematics, noting that, "learning new mathematics is an important part of expert practice for professional mathematicians" (p. 23). Samkoff, Lai, and Weber (2011) examined how mathematicians use diagrams to construct proofs.

This paper continues the above research's common thread of examining mathematicians' practices in doing and learning mathematics, in particular, in constructing proofs. Its findings should be useful in teaching proof construction in a way that complements prior research on university students' proving including: difficulties they encounter during the proving process (Moore, 1994; Weber & Alcock, 2004), difficulties with validations of proofs (Selden & Selden, 2003), and difficulties with comprehension of proofs (Conradie & Frith, 2000; Mejia-Ramos, et al., 2010) as well as Harel and Sowder's (1998) categorization of students' proof schemes, that is, the ways they decide what is true.

In analyzing mathematicians' proof construction practices, this paper focuses on impasses, as well as incubation and insight. These ideas have been used in the computer science, psychology, creativity, and mathematics education literatures, mainly in analyzing problem solving. A brief discussion of this literature will provide a background for this paper's usage of the terms in analyzing proof construction. Duncker (1945) defined an impasse as a "mental block against using an object in a new way that is required to solve the problem." He stated that, "[real] problem solving starts when a solver comes to an impasse." In contrast, Van Lehn (1990) appears to mean something different by an impasse. In his work on multi-digit subtraction, he described four categories of impasses in the execution of procedural knowledge. Those impasses were categorized by differences in the actions leading to the impasse, and were treated like "computer bugs."

Some computer scientists concerned with automatic theorem provers have a different meaning for (machine) impasses. Meier and Melis (2006) pointed out that an automatic theorem prover "gets stuck" when the computer has no further techniques with which to solve the current problem and ceases its pursuit of a proof. The actions programmed to attempt to overcome such impasses include building "proof plans," but even such plans have their limitations because "some proofs contain parts that are unique to that proof" (Lowe, Bundy, & McLean, 1998). Meier and Melis (2006) mentioned an advantage that humans have over automatic theorem provers: "When an expected progress does not occur or when the proof process gets stuck, then an intelligent problem solver [i.e., a person] analyzes the failure and attempts a new strategy."

One way human problem solvers sometimes recover from an impasse is through incubation. Incubation, according to Wallas (1926), is the process by which the mind goes about solving a problem, subconsciously and automatically. It is the second of Wallas' four stages of creativity:

- preparation (thoroughly figuring out what the problem is),
- incubation (when the mind goes about solving a problem subconsciously and automatically),
- illumination (receiving an idea after the incubation process), and
- verification (figuring out if the idea is correct).

In later work, Smith and Blankenship (1991) stated that "the time in which the unsolved problem has been put aside refers to the *incubation time*; if [illumination] occurs during this time, the result is referred to as an *incubation effect*" (p. 61). It has been conjectured that this effect happens best when one takes a break from creative work (Krashen, 2001). Illumination is also referred to by some authors as *insight*, or as a "Eureka" or "Aha!" moment (Bowden, Jung-Beemen, Fleck, & Kounios, 2005). Of such moments, Beeftink, van Eerde, and Rutte (2008) observed: "Individuals suddenly and unexpectedly get a good idea that brings them a great step further in solving a problem." In the neuroscience literature, Christoff, Ream, and Gabrieli (2004) have noted that insight, which might appear to be a spontaneously occurring thought process, "share[s] executive and cognitive mechanisms with goal-directed thought".

Both incubation and insight have been studied in psychology with mixed success. According to Smith and Blankenship (1991), "Several empirical studies have tested incubation effects in problem solving. A few of these experiments found incubation effects…In sum, these studies provide neither a strong base of empirical support for the putative phenomenon of incubation nor a reliable means of observing the phenomenon in the laboratory" (p. 62). More recently, Sio and Ormerod (2009), in their meta-analysis of 29 articles covering 117 separate experiments dealing with incubation, concurred: "Although some researchers have reported increased solution rates after an incubation period, others have failed to find effects" (p. 94). Psychologists have tried to provide better ways of capturing the creative process, including reinterpreting several theories of incubation (Helie & Sun, 2010), but have yet to provide consistent concrete evidence of success or failure.

To date the research on incubation in the mathematics education research literature has been sparse and primarily anecdotal. Byers (2007), in his view of creativity in mathematics, described stages similar to those of Wallas. In his investigation of mathematicians' practices, Hadamard (1945) mailed surveys to mathematicians around the world to develop his own ideas of what mathematicians do. More recently, in his dissertation research on AHA! experiences, Liljedahl (2004) used interviews with mathematicians to obtain data on insight. Liljedahl had tried creating an environment for mathematicians to exhibit insight, but conceded: "A further flaw in my experimental design was the role that the environment and setting play in the facilitation of AHA! experiences. Upon reflection, I now see that the clinical interview is not at all conducive to the fostering of such phenomena…" (p. 49). Still, both Hadamard and Liljedahl uncovered some evidence that mathematicians use incubation and then experience insights when solving problems. I hope to add to this literature, partly through narrowing the focus to theorem proving, making observations in a realistic setting, and supplying notes on an unfamiliar topic for the mathematicians to work on.

Theoretical Framework

By an *impasse*, I mean a period of time during the proving process when a prover feels or recognizes that his or her argument has not been progressing fruitfully and that he or she has no new ideas. What matters is not the exact length of time, or the discovery of an error, but the prover's awareness that the argument has not been progressing and requires a new direction or new ideas. Mathematicians themselves often colloquially refer to impasses as "being stuck" or "spinning one's wheels." This is different from simply "changing directions," when a prover decides, without much hesitation, to use a different method, strategy, or key idea, and the argument continues.

There appear to be two main kinds of mental or physical actions that provers take to recover from an impasse. One kind of action relates directly to the ongoing argument. The other kind of action consists of doing something unrelated which can be either mathematical or non-mathematical. Examples of both kinds will be provided below in the "Results" section.

While the treatments of impasses, incubation, and insight mentioned in the section on "Background Literature" may be useful in investigating a wide view of creativity and problem solving, constructing proofs in mathematics seems to be a topic that calls for some modification of the ideas. For example, all of the 117 experiments considered by Sio and Ormerod (2009) in their meta-analysis of incubation studies used an incubation period of just 1-60 minutes, but mathematicians routinely take more time to overcome impasses in their research, and their proofs tend to be rather long and complex. With this in mind, I define *incubation* as a period of time, following an attempt to construct at least part of a proof, during which similar activity does not occur, and after which, an insight (i.e., the generation of a new idea moving the argument forward) occurs. There might be ultimate success or failure with an insight arising from incubation, but that can only be determined by subsequent verification of the new idea's usefulness. A long proving process might entail several impasses and a number of incubation periods (and subsequent insights), only some of which ultimately contribute to the final proof.

New Data Collection Technique

Nine mathematicians (three algebraists, three topologists, two analysts, and one logician) agreed to participate in this study on proving. They were provided with notes on semigroups (Appendix A) containing 10 definitions, 7 requests for examples, 4 questions to answer, and 13 theorems to prove. The notes were a modified version of the semigroups portion of the notes for a Modified Moore Method course for beginning graduate students. This topic was selected because the mathematicians would hopefully find the material easily accessible, and because there are two theorems towards the end of the notes (Theorems 20 and 21 of Appendix A) that have caused substantial difficulties for beginning graduate students. During their exit interviews, two mathematicians offered that the choice of semigroups had been judicious, because they had been able to grasp the definitions and concepts quickly, and because at least one of the theorems had been somewhat challenging to prove. The data collection was split into two groups: four mathematicians writing proofs on tablet PCs, and five mathematicians writing proofs with a LiveScribe pen and special paper.

Tablet PC

With the tablet PC group, I approached each mathematician separately to explain how to use the hardware and the software. I explained how to use the stylus that came with the tablet PC and how to turn the tablet PC around in order to be able to write on it. There were two software programs on the tablet PC that the mathematicians were to work with: CamStudio screen-capturing software and Microsoft OneNote, which was the space on which the mathematicians wrote their proof attempts. The mathematicians each kept the tablet PC for a period of 2-7 days. After the tablet PC was returned, I analyzed the screen captures (resembling small movies in real time) and the mathematicians' proof writing attempts. All proof writing attempts on OneNote were exported as PDFs for analysis. One or two days after this initial analysis of a mathematician's work, I conducted an exit interview, during which I asked about their proofs and proof-writing (Appendix B).

LiveScribe Pen

The LiveScribe pen group consisted of five mathematicians. I approached each of these mathematicians separately to explain how to use the LiveScribe pen and special paper. The LiveScribe pen captures both audio and real-time writing using a camera near end of the ballpoint pen. When one presses on the "record" square at the bottom of the special paper with the pen, the pen goes into audio record mode, which then allows for the real-time capturing of the writing and speaking. The pen can be stopped by a "stop" button, and all proving periods are time-and-date stamped. Uploading the pen data to a computer goes through the LiveScribe software, and I exported each mathematician's collected proving periods together in one PDF file called a "pencast." The mathematicians each kept the LiveScribe pen and paper for a period of 1-10 days. I collected the work of each mathematician, analyzed the data for a period of 1-2 days, and then conducted an exit interview with each of them. The questions for this group of mathematicians were the same as those for the first group. Questions can be viewed in Appendix B.

Transition from Tablet PC to LiveScribe Pen

The switch from tablet PC to LiveScribe pen was done for several reasons. First, the tablet PC cost $900 and up, whereas the LiveScribe pens are just $99 and up. Second, the size of a movie file for a tablet PC screen capture of 16 minutes is one gigabyte, whereas an almost five hour proving session on the LiveScribe pen is just 60 megabytes. Third, the mathematicians were much more comfortable with pen and paper than with the tablet PC and a stylus, because they had to learn how to handle the tablet. Fourth, there were no visual or auditory quality differences between the data collected using the two techniques. This allowed for a smooth transition of data collection techniques to one that I felt was the most comfortable for the participants, and provided all the real-time data collection that I needed.

Summary Data

Four of the nine mathematicians that participated in the study had problems with the technology and thus did not produce “live” data. However, all four provided fixed written data, whether it was with the tablet on OneNote or writing on the LiveScribe paper without audio/video recording. From this data I could still conclude that some mathematicians had impasses because they were candid in writing all of their work, including crossing out failed attempts. The average total work time on the technology was two hours and five minutes. This time was calculated by adding the durations of their actual work, obtained from the date and time stamps. The average time from the first “clocked in” time-and-date stamp until the last “clocked out” time-and-date stamp was 19 hours, 56 minutes. The average number of pages written was slightly under 13. These three statistics allow one to conclude that the mathematicians expended considerable effort on the problems. Six of the nine mathematicians had impasses with one of the last two theorems. Most mathematicians worked through most of the theorems very quickly until they got to those final two theorems. Two of these mathematicians will be discussed in detail in the next section.

Results

Here is a description of an impasse, an incubation, and an insight leading to a proof for two of the mathematicians: Dr. A, an applied analyst and Dr. B, an algebraist. The technology worked for Dr. A and part of his work is described using the time-and-date stamps. For Dr. B, the technology did not work well, but good quality fixed written work and the exit interview data allow some of his work to be presented below in paragraph form.

Dr. A

In proving Theorem 21, "If S is a commutative semigroup with minimal ideal K, then K is a group," Dr. A experienced an impasse, an incubation, and a resulting insight. The following abbreviated, interpreted timeline illustrates this.

3:48 PM	7/13/11	9 min.	At this time Dr. A first attempted a proof of Theorem 21. He stopped and moved on to Question 22.
4:01 PM		16 min.	Continuing later, when he had finished Question 22, Dr. A

			scrolled up to his first proof attempt. He looked at his answer to Question 22, and at the ten minute mark, erased his first proof attempt. He then scrolled back to his proof of Theorem 20, viewed it for one minute, and wrote “the argument above proves that K has a multiplicative identity in S.” There was a brief pause, after which he scrolled up to the proof of Theorem 20 again for the final 30 seconds. Proving ended for the day at 4:17.
11:07 AM	7/14/11	11 min.	The next day Dr. A again started attempting to prove Theorem 21. But this time he used a mapping ϕ that multiplied each element by a fixed k_0 (an idea from his own research). He struggled with some computations until the end of this “clocked in” period.
11:32 AM		5 min.	When he “clocked in” again, Dr. A again worked with the mapping idea and then wrote, “I don’t know how to prove that K itself is a group. For example, I don’t know how to show that there is an element of K that fixes k_0,” acknowledging that he was at an **impasse**.
11:38 AM		23 min.	However, Dr. A continued trying unsuccessfully to use his mapping idea.
12:22 PM		6 min.	When Dr. A “clocked in” again, he continued trying unsuccessfully to use his mapping idea. For example, he wrote, “To prove ϕ is well-defined, let $tk_0 = x, tk_1 = k_2$. Let v be any other element of S such that $vk_0 = x$. Choose any $w \in S$ s.t. $wx = k_2$. Then $vk_1 = vwx = vwtk_0 = twvk_0 = twx = tk_1$. So $\phi(t)$ is determined once tk_0 is determined.”
12:55 PM	7/14/11	5 min.	Later on, when he “clocked in” again, after a 33-minute gap (which might be considered an **incubation** period), Dr. A proved Theorem 21 writing “Proof of theorem: We just need to show that K itself has no proper subideals. But K is principally generated, i.e., fix any $k_0 \in K$ and $K = \{sk_0 : s \in S\}$ since K is [a] minimal [ideal]. If L were a proper ideal of K…” Notice that this idea (an **insight**) for proving Theorem 21 differs from the idea he had tried 33 minutes earlier.

Dr. A indicated in his exit interview where he had had an impasse, noting "One has to show there aren't any sub-ideals of the minimal ideal itself, considered as a semigroup, and that's where I got a little bit stuck." This is because the concept of ideal really depends on the containing semigroup, here S or K. Dr. A also indicated how he consciously generally recovers from impasses: he prefers to get "un-stuck" by walking around, but distractions caused by his departmental duties also help. That is, he often takes a break from his creative work by purposely doing something unrelated. In this case, Dr. A took several such breaks, but only the last one yielded a new idea.

Dr. B

Dr. B experienced an impasse on the penultimate theorem (Theorem 20), "If S is a commutative semigroup with no proper ideals, then S is a group." Unfortunately with Dr. B, there were no screen captures, but his written proof attempts were very detailed and the exit interview was very informative. He wrote, "Stuck on [Theorem] 20. It seems you need [to hypothesize] $1 \in S$, but I can't find a counterexample to show this [that the theorem is false]." Dr. B next moved on to the final theorem (Theorem 21), the one on which Dr. A had had an impasse, proved it correctly, and then crossed out his proof, probably because he had used his as yet unproved Theorem 20. After that, he moved on to the final request for examples (Question 22), explaining in his exit interview, "I moved on because I was stuck [on Theorem 20]...maybe I was going to use one of those examples...I might get more information by going ahead." Dr. B's next approach was to attempt to create counterexamples for Theorem 20. After considering his candidates for counterexamples for some time and being interrupted by taking his family to lunch, Dr. B proved both theorems correctly.

In his exit interview, Dr. B stated that he had developed a belief that had confused him, and thought that he needed to assume that there was an identity element. He also said, "I probably spent 30 minutes to an hour trying to come up with a crazy example. I went to lunch and while I was at lunch, then it occurred to me that I was thinking about it the wrong way. So I went back then and it was quick [using that insight]."

Impasse Recovery

Below are descriptions of the various actions the mathematicians in this study used to recover from impasses, listed according to whether they were directly related to the ongoing argument, or not directly related to it.

Some actions were observed in the proving processes of the mathematicians in the data collected while they worked alone, whereas other actions were first mentioned during the exit interviews and focus group discussions. All of the described actions have exit interview or focus group quotes from the mathematicians explaining them.

Impasse recovery actions that are directly related to the argument

(a) Using methods that occurred earlier in the session: Some of the mathematicians in this study tried to use a proving technique that they had used earlier in the proving session to overcome an impasse.

> "It would be fairly easy to prove…it's likely an argument, kind of like the one I already used…" (Dr. H)

(b) Using prior knowledge from their own research: There were mathematicians in this study who tried to use their own research to overcome an impasse.

> "I'm trying to think if there's anything in the work that I do that...I mean some of the stuff I've done about subspaces of $L^2(\mathbb{R})$, umm...there are things called principal shift invariance spaces that the word principal comes into play." (Dr. A)

(c) Using a (mental) database of proving techniques: One of the mathematicians, Dr. F, had a (mental) database of proving techniques in her head.

> "Your brain is randomly running through arguments you've seen in the past… standard techniques that keep running through my head, sort of like downloading a whole bunch at the same time and figuring out which way to go." (Dr. F)

(d) Doing other problems in the problem set and coming back to the impasse: Five of the nine mathematicians in the study approached their proving impasses by moving on to consider the rest of the problems in the notes.

> "I moved on because I was stuck...maybe I was going to use one of those examples ... I might get more information by going ahead." (Dr. B)

(e) Generating examples or counterexamples: Three of the mathematicians in the study attempted to construct counterexamples to some of the theorems when they felt a theorem had not been correctly stated.

> "At first I thought, 'How could I prove this?' And I didn't immediately think of a proof. Then I thought, 'what about a counterexample?' and pretty quickly I came up with a counterexample, of course which turns out not to be right." (Dr. G)

Impasse recovery actions that are unrelated to the argument

(a) Doing other mathematics: Some mathematicians indicated that they might go to another project to help them overcome proving impasses.

> "What I try to do is to keep three projects going…I make them in different areas and different difficulty levels…" (Dr. E)

(b) Walking around: Some mathematicians indicated that sometimes they may choose to walk around to overcome a proving impasse.

> "When I'm stuck, I often feel like taking a break. And indeed, you come back later and certainly for a mathematician you go off on a walk and you think about it." (Dr. G)

(c) Doing tasks unrelated to mathematics: This is the second non-mathematical action unrelated to an impasse. This action was also perhaps the most unusual, and Dr. E seemed slightly embarrassed when he reported the action to me.

> "Yeah I'll do something else, and I'll just do it, and if there's a spot where I get stuck or something, I'll put it down and I'll watch TV, I'll watch the football game, or whatever it is, and then at the commercial I'll think about it and say yeah that'll work…" (Dr. E)

(d) Going to lunch/eating: This action was shown to be effective with Dr. B.

> "So I had spent probably the last 30 minutes to an hour on that time period working on number [Theorem] 20 going in the wrong direction. Ok, so I went to lunch, came back, and while I was at lunch, I wasn't writing or doing things, but I was just standing in line somewhere and it [an insight] occurred to me the…(laughs)…how to solve the problem." (Dr. B)

(e) Sleeping on it: The last action to overcome an impasse seems to be the easiest for a mathematician. Proving can involve mental exhaustion, so resting can help one's exploration for new ideas.

> "It often comes to me in the shower…you know you wake up, and your brain starts working and somehow it [an insight] just comes to me. I've definitely gotten a lot of ideas just waking up and saying "That's how I'm going to do this problem."
> (Dr. F)

The first of the above actions, namely, doing other mathematics, is mathematical, whereas the remaining actions are non-mathematical diversions. Most of the actions that the mathematicians took to overcome their proving impasses were enacted more or less automatically and were not mentioned during their proving sessions. However, the mathematicians did acknowledge those actions during their exit interviews or in the focus group discussions.

Discussion

A majority of the nine mathematicians in this study exhibited impasses and recoveries from those impasses, including some due to incubation. Furthermore, there were a number of instances in which impasses and recoveries, or incubations, might have occurred in a way that could not be observed. For example, all of the mathematicians reported that when they first received the notes they immediately read them to estimate how long the proofs might take, but none started proving right away. In addition, there were periods during the proving sessions when nothing was recorded, and there were also substantial gaps between the “clock in” and “clock out” times during the proving sessions. Furthermore, when the mathematicians next “clocked in” after having left a proof attempt without finishing it, they almost always had a new idea to explore.

In the focus groups, the mathematicians also discussed methods of impasse recovery and what amounts to incubation (that can occur independent of an impasse). They all did this in a relaxed, assured way, not like someone discussing something unfamiliar, but rather like someone discussing beliefs built up over some time. They described a remarkable number of ways of recovering from an impasse. Furthermore, they mentioned benefits that appear to go beyond just restarting an argument.

During one focus group interview, Dr. G stated, “When we are working on something, we are usually scribbling down on paper. When you go take a break,… you are thinking about it in your head without any visual aids….[walking around] forces me to think about it from a different point of view, and try different ways of thinking about it, often global, structural points of view.” There is no “scribbling on paper.” Doing this, he believed, might assist in understanding the structure of a problem or even of an area of mathematics. In a somewhat similar vein, Dr. F offered the following, “You just come back with a fresh mind. [Before that] you’re zoomed in too much and you can’t see anything around it anymore.” This seems to be a somewhat more local broadening perspective.

From Dr. G, one sees that there might still be conscious thought about the current mathematical problem going on during a break so he is not referring just to incubation. Dr. F added that “freshness” of mind might also help with overcoming proving impasses. Also, simply going away from and coming back to a problem or proof might yield new ideas for recovering from an impasse. Dr. A stated, “I do have a belief that if I walk away from something and come back it’s more likely that I’ll have an idea than if I just sit there.” These remarks indicate that some mathematicians take deliberate actions to overcome impasses and also to improve the breadth or quality of their perspectives.

Conscious, or deliberate, incubation has been shown in the psychology literature to result in a greater incubation effect than merely being interrupted during the problem-solving process. “Individuals who took breaks at their own discretion (a) solved more problems and (b) reached fewer impasses than interrupted individuals” (Beeftink, van Eerde, & Rutte, 2008). Ironically, interruption seems to have been useful in the case of Dr. B, who said that he would have worked non-stop if he had not been interrupted for lunch with his family. This also agrees with the psychology literature: “It was also found that interrupted individuals reached fewer impasses than individuals who worked continuously on problems” (Beeftink, van Eerde, & Rutte, 2008).

Educational Implications

The Results and Discussion sections above suggest that proving impasses, recoveries from them, incubation, insight, and the ability to deal with such topics is a significant part of doing mathematics, and in particular, of constructing proofs. Thus, it is worth examining how they might be taught. The ways of doing this are yet to be examined in detail, however, one small example can be provided. The professors in this study were unaware of the origin of the notes (Appendix A), and one tried to construct counterexamples. In fact, the notes were designed for teaching beginning graduate students about proving. Theorems 20 and 21 can be made much easier by adding a comment about careful reading of the definition of ideal (Definition B of Appendix A) and by adding two easily proved lemmas for Theorem 20. These were omitted from the notes to provide students with experiences similar to those of these professors'. Most students would probably require several attempts and some advice for proving Theorems 20 and 21. However, the experience of trying may still be valuable.

Similar experiences can probably be provided to students who are not yet familiar with constructing proofs by considering problem solving. A problem that is likely to generate impasses is probably close to what Schoenfeld (1982) described as a "rich" problem:

- The problem needs to be accessible. That is, it is easily understood, and does not require specific knowledge to get into.
- The problem can be approached from a number of different ways.
- The problem should serve as an introduction to important mathematical ideas.
- The problem should serve as a starting point for rich mathematical exploration and lead to more good problems (as cited by Liljedahl, 2004, pp. 187-188).

Notice that in the list of actions to overcome impasses, the mathematicians moved on to consider the request for examples (Question 22), having observed that considering them might be useful. This action to overcome an impasse relates well to students' experiences, because homework assignments usually consist of multiple problems, so they can go ahead to another problem when they are "stuck." Furthermore, students may need to experience successes in order to acquire confidence in their proving ability, and telling them what mathematicians do when they "get stuck" might help them when they have "no idea what to do next." Moreover, there is encouragement from the psychology literature about the positive effects of incubation in the classroom. Sio and Ormerod (2009) listed four articles where "educational researchers have tried to introduce incubation periods in classroom activity, and positive incubation effects in fostering students' creativity have been reported." (p. 94)

Future Research

Using LiveScribe pens and the corresponding paper provides a naturalistic setting for provers while gathering real-time data from them. If one can see what a mathematician does during the proving process, those same techniques might be used with students in a transition-to-proof or proof-based course. How can we use this data collection technique in the classroom? Will it benefit students to have LiveScribe pens with which to do their homework so that teachers can analyze their proving processes?

How can we gain additional information on when and how incubation is used in mathematics by mathematicians or students? How can we collect more of the actions that mathematicians use to recover from impasses? Also, how can we encourage students to take some of these actions to recover from their proving and problem-solving impasses?

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Appendix A

Definition A: A semigroup $(S,\cdot)$ is a nonempty set S together with a binary operation $\cdot$ on S such that the operation is associative. That is, for all $a, b,$ and $c \in S$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
Note: We often refer to the "semigroup S" instead of the "semigroup $(S,\cdot)$" and symbols such as $+, -, *, \odot, \oplus$ may be used instead of "$\cdot$". Also, "$a \cdot b$" or "ab" is often read "a times b".

Example 1: Find several examples of semigroups.

Definition B: A nonempty subset A of a semigroup S is called a left ideal [right ideal, ideal] of S if $SA \subseteq A$ [$AS \subseteq A, AS \cup SA \subseteq A$] where $SA = \{sa | s \in S$ and $a \in A\}$.

Example 2: Find some examples of left ideals, right ideals, and ideals in several semigroups.

Theorem 3: The intersection of a left ideal and a right ideal is nonempty.

Theorem 4: The intersection of two ideals is an ideal.

Definition C: A non-empty subset T of a semigroup S is called a subsemigroup of S if $TT \subseteq T$.

Note: In a semigroup, every left ideal, right ideal, and ideal is a subsemigroup.

Definition D: A semigroup S is called commutative or Abelian if, for each a and $b \in S$, $ab = ba$.

Example 5: Find some ideals in [0,1] under multiplication.

Definition E: An element e of a semigroup S is called an idempotent if $ee = e$. ("ee" is often written "e^2".)

Definition F: An element 1 of a semigroup S is called an identity element of S if, for each $s \in S$, $1s = s1 = s$. (Other symbols, such as "e", may be used instead of "1" to represent an identity element.)

Definition G: An element 0 of a semigroup S is called a zero element of S if, for each $a \in S$, $0a = a0 = 0$. (Other symbols may be used instead of "0" to represent a zero element.)

Example 6: Find a semigroup with an idempotent which is neither the identity nor a zero.

Definition H: An ideal [left ideal, right ideal] K of a semigroup S which does not properly contain any other ideal [left ideal, right ideal] of S is called a minimal [left, right] ideal of S.

Example 7: Find some semigroups that contain, and some that do not contain, a minimal ideal.

Question 8: Can a semigroup be its own minimal ideal?

Theorem 9: Every semigroup has at most one minimal ideal.

Example 10: Find examples of semigroups that (1) are not commutative, (2) do not have idempotents, and (3) consist entirely of idempotents.

Theorem 11: A semigroup can have at most one identity element and at most one zero element.

Theorem 12: Distinct minimal left [right] ideals of a semigroup are disjoint.
Note: If a semigroup has a minimal ideal, it is unique (by Theorem 9) and it is called the kernel of the semigroup. The theory of semigroups started (in 1928) when Suschewitsch characterized the kernel.

Definition I: Let S and T be semigroups and $f: S \to T$ be a function. We call f a homomorpism if, for each $x \in S$ and $y \in S$, $f(xy) = f(x)f(y)$. If f is also one-to-one, f is called an isomorphism. We say S and T are isomorphic if f is an onto isomorphism.

Note: We think of semigroups S and T as the "same" if there is an onto isomorphism $f: S \to T$.

Example 13: Find some examples of homomorphisms that are, and are not, isomorphisms. Also find some examples that are, and are not, onto.

Theorem 14: Let S and T be semigroups and $f: S \to T$ be a homomorphism. If $e \in S$ is an idempotent, then $f(e)$ is an idempotent.

Theorem 15: Let S and T be semigroups and $f: S \to T$ be a homomorphism. If A is a subsemigroup of S, then $f(A)$ is a subsemigroup of T.

Theorem 16: Let S and T be semigroups and $f: S \to T$ be an onto homomorphism. If e is an identity [zero] of S, then $f(e)$ is an identity [zero] of T.

Theorem 17: Let S and T be semigroups and $f: S \to T$ be an onto homomorphism. If I is an ideal of S, then $f(I)$ is an ideal of T.

Definition J: A semigroup G is called a group if G has an identity 1 and if for each $g \in G$ there is a $g' \in G$ such that $gg' = g'g = 1$.

Theorem 18: Let G be a group with identity 1. If $g, g', g'' \in G$ with $gg' = g'g = 1$ and $gg'' = g''g = 1$, then $g' = g''$. (That is, the element g' so that $gg' = g'g = 1$ is unique. The element g' is called the inverse of g in G and written g^{-1}.)

Theorem 19: A group has no proper left ideals [right ideals, ideals].

Theorem 20: If S is a commutative semigroup with no proper ideals, then S is a group.

Theorem 21: If S is a commutative semigroup with a minimal ideal K, then K is a group.

Question 22: For each of parts a, b, and c are the two semigroups isomorphic? Prove you are right.

(a) $(\mathbb{Z}, +)$ where $\mathbb{Z}$ is the integers and $+$ is ordinary addition. $(2\mathbb{Z}, +)$ where $2\mathbb{Z}$ is the even integers and $+$ is ordinary addition.

(b) $(\mathbb{R}, +)$ where $\mathbb{R}$ is the real numbers and $+$ is ordinary addition. $((0, \infty), \cdot)$ where $(0, \infty)$ is the positive real numbers and $\cdot$ is ordinary multiplication.

(c) $(L, \cdot)$ where $L = \{0,1,2,3,4\}$ and for $x, y \in L$, $x \cdot y = x$. $(\mathbb{Z}_5, \cdot)$ where $\mathbb{Z}_5 = \{0,1,2,3,4\}$ and $x \cdot y$ means "$xy \ mod \ 5$", i.e., ordinary multiplication minus (whole) multiples of 5. For example, $4 \cdot 4 = 16 - (3 \times 5) = 1, 3 \cdot 4 = 12 - (2 \times 5) = 2$, and $3 \cdot 3 = 9 - 5 = 4$, but $2 \cdot 2 = 4$.

Appendix B

Interview Questions

1. Was there anything that was particularly difficult or took you long?
2. (When there were delays) What were you thinking of at this point in time?
3. What made you think of ______ (e.g., stabilizer)?
4. What difficulties were there with the technology?
5. (With a very long delay, e.g., of several hours) What did you do in that time period? Did you think about the notes or some theorem in the notes?

Focus Group Questions

1. (Question to get them comfortable) What did you think of these notes?
2. Compare and contrast your experiences with the last 2 theorems.
3. If and when you did get stuck with these notes, how did you handle that?
4. In general, what do you do when you get stuck (in a problem, proof, with your research)?
5. Is there anything else you do or think about when attempting to prove theorems?