Proof and Proving: Logic, impasses, and incubation

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Outline of the talk

- Logic in student-constructed proofs
  - A question
  - A “chunk-by-chunk” analysis of student proofs
  - An answer
  - Logic-like structures
- Observing mathematicians proving alone
  - New data collection technique
  - Impasses – “getting stuck”
  - Incubation
- Future research

How much logic is in student proofs?

- Often university mathematics departments teach some formal logic early in a transition-to-proof course in preparation for teaching undergraduate students to construct proofs.
- There are some that believe that formal logic should be taught first, separately (Epp, 2003) and some that believe that logic need not be explicitly taught at all (Hanna & de Villiers, 2008).

How much logic is in student proofs? (cont.)

- One aim of the study was to find the logic beyond common sense in student-constructed proofs so that the question of how it should be taught can be better understood.
- If formal logic occurs quite a bit, then teaching a unit on predicate and propositional calculus first might be a good idea. However, if formal logic is infrequent, then teaching logic in context, while teaching proving, might be more effective.

The setting

- The proofs come from an "Understanding and Constructing Proofs" course at a large southwestern university.
  - This course was for advanced undergraduates and beginning graduates.
  - There were 42 theorems covering sets, functions, real analysis, algebra, and topology.
- Students constructed proofs at home, presented their proofs on the blackboard, and these were discussed.
- For each theorem, one proof was approved by the professor and copies were given to everyone in the class.

The coding

- In this study, I coded all the student-constructed proofs of theorems using a “chunk-by-chunk” analysis.
- There were several iterations of the coding process during which the categories of chunks emerged.
- One iteration included having two mathematics professors coding several theorems and meeting twice to discuss the coding.

Chunk-by-chunk analysis

- The “chunks” are somewhat similar to those used in analyzing short-term memory (Miller, 1956). They are small phrases that can be taken together as a “meaningful unit” in thinking.
- Some chunks can be sentences, others can be one or a couple of words, but they are always meant to refer to a moment or unit in the proof.
- The two professors and I were in agreement on over 80% of the chunks in 4 proofs during one chunking iteration.

Example of a chunk-by-chunk analysis

Theorem 3: For sets $A, B,$ and C, if $A \subseteq B$ then $C - B \subseteq C - A$.

Proof: Let A, B, and C be sets such that $A \subseteq B$. Suppose $x \in C - B$. Then $x \in C$ and $x \notin B$. By $A \subseteq B$ we have $x \notin A$; hence $x \in C - A$. Therefore, $C - B \subseteq C - A$.

1. Let A, B, and C be sets
2. such that $A \subseteq B$.
3. Suppose $x \in C - B$.
4. Then $x \in C$ and $x \notin B$.
5. By $A \subseteq B$
6. we have $x \notin A$;
7. hence $x \in C - A$.
8. Therefore, $C - B \subseteq C - A$.

The categories

- During the coding of the chunks, 13 categories emerged.
- I will describe five of the categories; two about logic and the three that occurred most often.

Five of the categories

- Informal Inference (common sense) (**II**)
  - Can be made without bringing to mind formal logic (by students at the beginning of a transition-to-proof course)
  - A common example is modus ponens
- Formal Logic (**FL**)
  - Inference requiring predicate or propositional calculus of the kind taught in a transition-to-proof course, and not informal inference
  - Beginning transition-to-proof students might not know this “formal logic”
  - An example would be: if $x \notin B \cup C$, then $x \notin B$ and $x \notin C$

Five of the categories (cont.)

- Definition (**DEF**)
  - The chunk is immediately derived from the definition
- Assumption (**A**)
  - Introducing a mathematical object or assuming properties of the object
  - Two Sub-categories
    - Example: For the theorem "For all $n \in \mathbb{N}$, if $n > 5$ then $n^2 > 25$."
      - Choice (**A-C**): "Let $n \in \mathbb{N}$"
      - Hypothesis (**A-H**): "Suppose $n > 5$"
- Interior Reference (**IR**)
  - Referring to a chunk or chunks stated earlier in the proof

Example

- Theorem 38: If X is a Hausdorff space and $x \in X$, then $\{x\}$ is closed.
- Proof: Let X be a Hausdorff space. Let $x \in X$. Note $\{x\} = X - (X - \{x\})$. Suppose $y \in X$ and $y \neq x$. Because X is Hausdorff, there is an open set P_y for which $y \in P_y$. There is also an open set R_y such that $x \in R_y$ and $P_y \cap R_y = \emptyset$. Suppose $P_y \nsubseteq X - \{x\}$, then $x \in P_y$, but $x \in R_y$. Therefore $x \in P_y \cap R_y$, which is a contradiction. Therefore, $P_y \subseteq X - \{x\}$. Thus for every $y \neq x$ there is an open set P_y where $y \in P_y$ and $P_y \subseteq X - \{x\}$. The union of all P_y is equal to $X - \{x\}$, which is thus an open set. Therefore $\{x\}$ is closed, being the complement of an open set.

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Example (cont.)

Let X be a Hausdorff space.	Assumption (Hypothesis)
Let $x \in X$.	Assumption (Hypothesis)
Note $\{x\} = X - (X - \{x\})$.	Formal Logic
Suppose $y \in X$ and $y \neq x$.	Assumption (Choice)
Because X is Hausdorff,	Interior reference
there is an open set P_y for which $y \in P_y$. There is also an open set R_y such that $x \in R_y$ and $P_y \cap R_y = \emptyset$.	Definition of Hausdorff
Suppose $P_y \nsubseteq X - \{x\}$,	Assumption (Hypothesis)
then $x \in P_y$,	Informal inference
but $x \in R_y$.	Interior reference
Therefore $x \in P_y \cap R_y$,	Definition of intersection
which is a contradiction.	Contradiction statement

Example (cont.)

Therefore, $P_y \subseteq X - \{x\}$.	Informal Inference
Thus for every $y \neq x$ there is an open set P_y where $y \in P_y$ and $P_y \subseteq X - \{x\}$.	Conclusion statement
The union of all P_y is equal to $X - \{x\}$,	Formal Logic
which is thus an open set.	Definition of topology
Therefore $\{x\}$ is closed, being the complement of an open set.	Conclusion statement/Definition of closed

Results

- In the 42 proofs, consisting of 673 chunks, formal logic (**FL**) constituted 1.9% of the chunks (13 chunks), while informal inference (**II**) was 6.5% (or 44 chunks).

- Definition (**DEF**): 30% of the proof chunks (or 203 chunks)
- Assumption (**A**): 25% of the proof chunks (166 chunks)
- Interior reference (**IR**): 16% of the proof chunks (108 chunks)

Why such a small percentage of formal logic?

- The course was intended to cover a wide variety of kinds of proofs, causing many of the proofs to be based mainly on definitions.
- The coding did not consider the *implicit* logical actions in the proving process or in the structuring of proofs.

Homology Class

- In Fall 2010, I took a Homology course and chose to code 10 proofs using the same categories.
- The proofs were from another student in the class who got a perfect score on all homework.
- I found that less than 1% of the 170 chunks could be coded as formal logic (**FL**), while informal inference (**II**) had 10%.
- Definition (**DEF**), assumption (**A**), and interior reference (**IR**) were the three highest percentages (21% vs. 30%, 18% vs. 24% and 17% vs. 16% respectively).

An answer

- From this study, one can see that there were few instances of formal logic (predicate or propositional calculus that are not common sense). This indicates that it may be more beneficial to teach logic while teaching proving.
- There are many instances in a proof where logic is being used implicitly (such as using a definition), and there are truth-preserving structures in proofs. These structures were not counted in this coding because they entail a global view.

Logic-like structures

- A logic-like structure preserves truth value in an argument, yet is not in the language of predicate or propositional calculus.
- For example, if one sees a situation where the theorem states “For all $x \in A$, $P(x)$”, one starts with “Let $x \in A$.” and reasons to “$P(x)$”
- Another example would be to prove there is a unique x so that $P(x)$, one starts with “Suppose $P(a)$ and $P(b)$” and reasons to “$a = b$”. This logic-like structure shows up in proving that an identity is unique in a semigroup.

Examining students' approaches to logic-like structures

- I videoed and interviewed 3 students from the "proofs" class one year later.
- 45 minutes were focused on the uninterrupted, think-aloud production of the proof, followed by 15 minutes of follow-up interview.
- One page of notes was given to the students starting with the definition of semigroup and supplying all information needed to prove the theorem.
- The theorem: Every semigroup has at most one minimal ideal.

Results

- No one finished the proof correctly after 45 min. One student finished, but with some gaps in her proof.
- Every student immediately considered a semigroup S, and all approached the proof by assuming two, or n, minimal ideals.
- After this, each student proved the theorem differently, but that did not mean more logic was used.

Motivation and Questions

- These 3 students should have been able to prove the theorem but could not in the 45 minute interview.
- All 3 “got stuck” during the interview.
- How can people be observed constructing proofs alone (with unlimited time)?
- Do mathematicians “get stuck” and how do they get “un-stuck?”

Background Literature

- Mathematicians’ knowledge
  - Actions during proof validations (Weber, 2008)
  - Mathematicians’ learning (Burton, 1999; Wilkerson-Jerde & Wilensky, 2011)
  - Using diagrams to construct proofs (Samkoff, Lai, & Weber, 2011)
- Students’ proving
  - Difficulties (Moore, 1994; Weber & Alcock, 2004)
  - Validations of proofs (Selden & Selden, 2003)
  - Comprehension of proofs (Conradie & Frith, 2000; Mejia-Ramos, et al., 2010)

Impasses

- Impasse – A period of time when a prover feels or recognizes the argument is not progressing and he or she has no new ideas
  - Also known as “getting stuck” or “spinning one’s wheels”
  - Different from an impasse defined for automated computer provers (Meier & Melis, 2005)
- Two kinds of actions to recover from an impasse
  - Mathematical or non-mathematical

Incubation

- Incubation – a period of time, following a proof attempt, during which similar activity does not occur
- The second stage of the 4 stages of creativity (Wallas, 1926)
  - Preparation, Incubation, Illumination, Verification
- Poincare, Hadamard, and other mathematicians have described a period of incubation, followed by an "insight"
- Apparently should have interest in finding the solution for incubation to have any effect

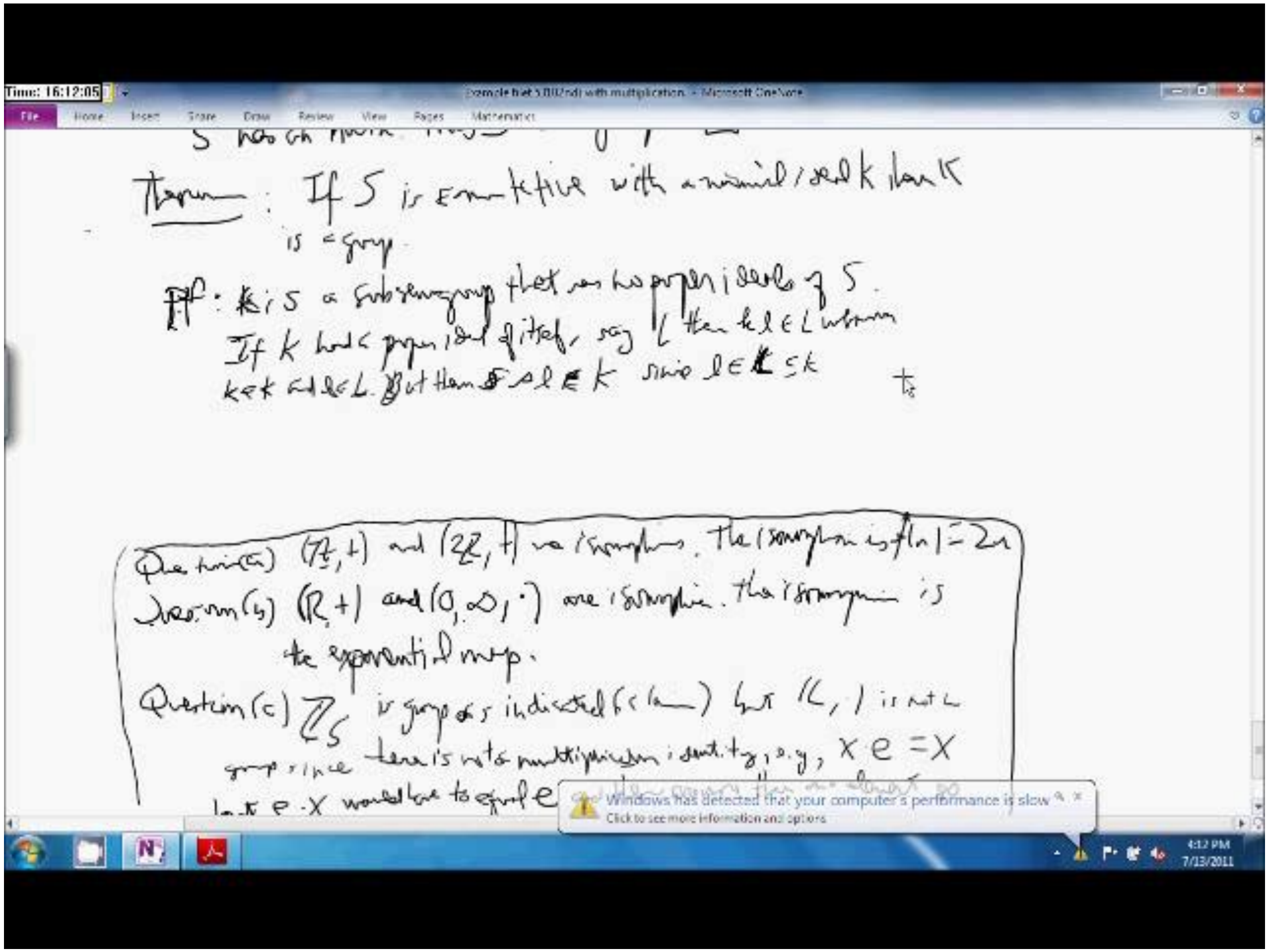
Participants and Tasks

- Nine research mathematicians (3 algebraists, 2 analysts, 3 topologists, 1 logician)
- Tasks – prove theorems in notes on semigroups (10 definitions, 13 theorems, 7 example requests, and 4 questions)
- Chosen for two reasons
  - Material (I hoped) was unfamiliar but accessible
  - Last two theorems require non-obvious lemmas and were difficult for students

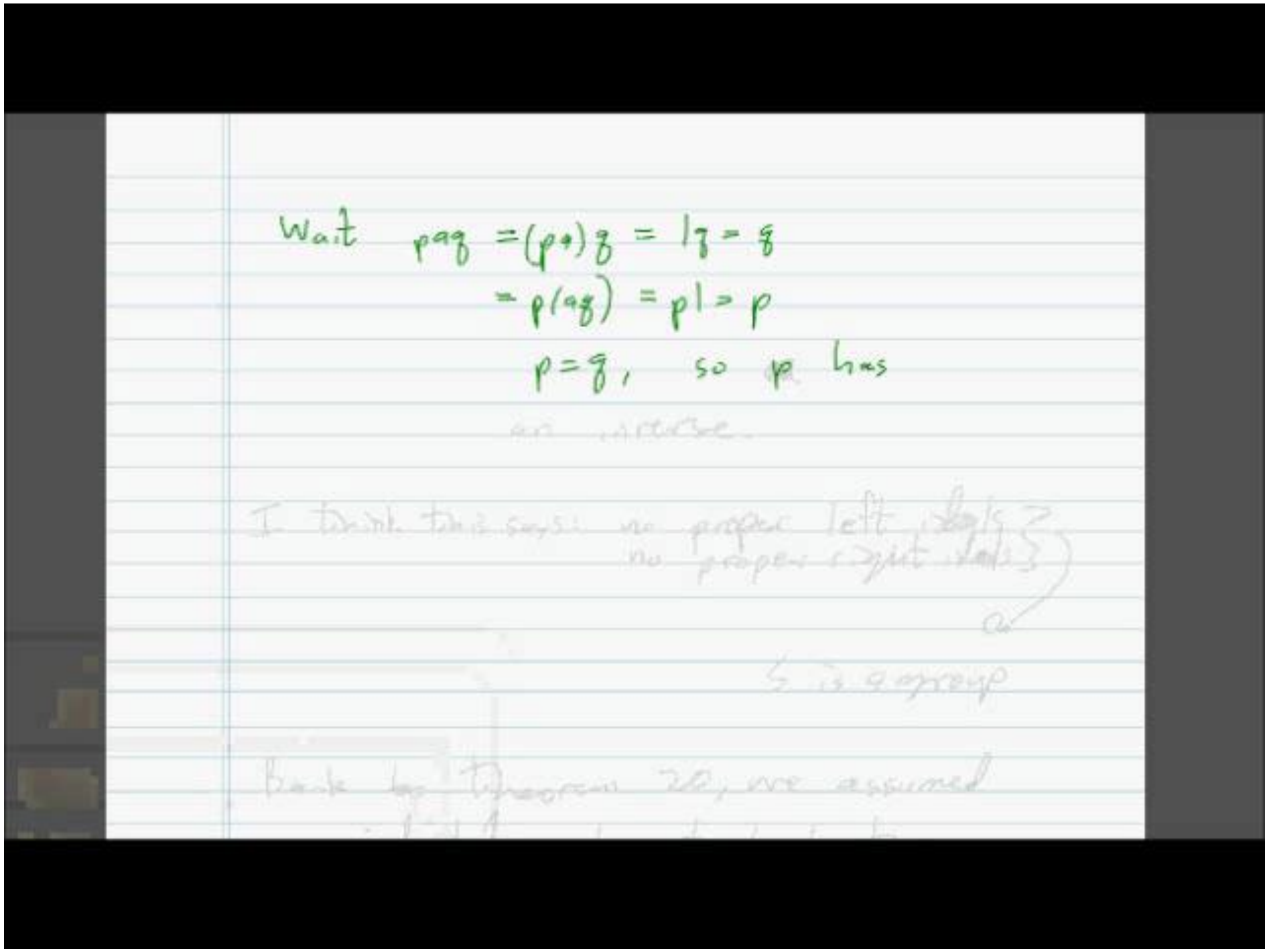
Data Collection

- Electronically:
  - The first four mathematicians proved on a tablet PC, set-up with CamStudio (screen-capturing software) and OneNote (space for their writing).
  - The final five mathematicians proved with a LiveScribe pen and special paper, capable of recording audio and writing in real-time.
- Both had date and time stamps for each writing session
- Advantages:
  - Used at the participant’s leisure
  - Real-time recording of the proving process
  - Never done before

Example of Tablet PC



Example of LiveScribe pen



Data Collection, cont.

- Each mathematician kept the equipment for 2-7 days.
- I analyzed the screen captures and the proof attempts.
- One or two days later, I interviewed the mathematicians about their proofs and their proving attempts.
- I also had two videoed “focus group” sessions: one for the tablet participants, the other for the LiveScribe pen participants.
- Two mathematicians volunteered the choice of semigroups was judicious:
  - Grasp concepts quickly
  - At least one of the theorems was challenging to prove

Summary Data

- 4 of the 9 professors had problems with the equipment, and thus did not produce “live” data
- 6 of the 9 professors had impasses with at least one of the last two theorems
- Average time of a professor’s work on the technology: 2 hours, 5 minutes
- Average time from first technology time stamp until the last: 19 hours, 56 minutes
- Average amount of pages written: Around 13

Dr. A

- Applied analyst
- Encountered impasse with the final theorem in the notes: “If S is a commutative semigroup with minimal ideal K, then K is a group.”
- Done on a tablet PC
- Total time: 22 hours, 17 minutes (July 13, 2:44 PM - July 14, 1:01 PM)

Proving Process of Dr. A, Day 1

- 3:48 PM Attempted a proof of Theorem 21 by contradiction
- 3:54 PM Moved on to the final part of the notes containing a request for examples
- 4:05 PM Scrolled on the screen back up to view his first proof attempt, which he then erased.
- 4:12 PM Attempted the proof again

Proving Process of Dr. A, Day 2

- Next screen capture at 11:07 AM of Day 2.
  - Used mappings and inverse mappings of elements
- “I don’t know how to prove that K itself is a group.”
- After a 30-minute gap, he proved the theorem successfully.

Dr. A's Exit Interview

- Dr. A acknowledged his impasse:
  - "One has to show there aren't any sub-ideals of the minimal ideal itself, considered as a semigroup, and that's where I got a little bit stuck."
- Dr. A gets out of this impasse (consciously) by walking around and doing his departmental duties.

Dr. B

- Algebraist
- Encountered impasse with the penultimate theorem in the notes: "If S is a commutative semigroup with no proper ideals, then S is a group."
- Done on a tablet PC
- Did not get any screen captures due to failure with software
- Total time: 5 hours, 40 minutes (August 3, 7:25 AM – August 3, 1:05 PM)

Proving Process of Dr. B

- Dr. B wrote: “Stuck on [theorem] 20. It seems you need $1 \in S$ [in the hypothesis], but I can't find a counterexample to show this.”
- Moved on to the next theorem, which he proved correctly, but then struck out his work.
- Then Dr. B went to the final question dealing with examples of isomorphisms of semigroups.
- Dr. B was interrupted to go to lunch.
- After lunch, Dr. B proved both theorems correctly.

Dr. B's Exit Interview

- Dr. B stated that he had created a property that had confused him, and thought that he needed to assume that there was an identity.
  - "I probably spent 30 minutes to an hour trying to come up with a crazy example."
- He said he got out of his impasse by going to lunch with his family, noting that he would have worked on the problem continuously if not for the lunch.

Actions to Overcome Impasses

- Viewing the impasses, the action to overcome utilizes mathematical actions or does not
- Hence, the actions are separated into two categories:
  - Mathematical
  - Non-mathematical
- All the actions to overcome impasses are accompanied by exit interview quotes from professors supporting the action

Mathematical Actions

- Using methods that occurred earlier in the session
  - “It would be fairly easy to prove…it’s likely an argument, kind of like the one I already used…” (Dr. H)
- Using prior knowledge from their own research
  - “I'm trying to think if there's anything in the work that I do that...I mean some of the stuff I've done about subspaces of $L^2(\mathbb{R})$, umm...there are things called principal shift invariance spaces that the word principal comes into play.” (Dr. A)

Mathematical Actions, cont.

- Using a database of proving techniques
  - “Your brain is randomly running through arguments you’ve seen in the past… standard techniques that keep running through my head, sort of like downloading a whole bunch at the same time and figuring out which way to go.” (Dr. F)
- Doing other problems and coming back to their impasse
  - “I moved on because I was stuck...maybe I was going to use one of those examples...I might get more information by going ahead.” (Dr. B)

Mathematical Actions, cont.

- Doing other mathematics
  - “What I try to do is to keep three projects going…I make them in different areas and different difficulty levels…” (Dr. E)

Non-Mathematical Actions

- Walking
  - “When I’m stuck, I often feel like taking a break. And indeed, you come back later and certainly for a mathematician you go off on a walk and you think about it.” (Dr. G)
- Watch TV
  - “Yeah I’ll do something else, and I’ll just do it, and if there’s a spot where I get stuck or something, I’ll put it down and I’ll watch TV, I’ll watch the football game, or whatever it is, and then at the commercial I’ll think about it and say yeah that’ll work…” (Dr. E)

Non-mathematical Actions, cont.

- Going to lunch/eating
  - “So I had spent probably the last 30 min to an hour on that time period working on number 20 going in the wrong direction. Ok, so I went to lunch, came back, and while I was at lunch, I wasn’t writing or doing things, but I was just standing in line somewhere and it occurred to me the…(laughs)…how to solve the problem.” (Dr. B)
- Waking up
  - “It often comes to me in the shower…you know you wake up, and your brain starts working and somehow it just comes to me. I’ve definitely gotten a lot of ideas just waking up and saying “That’s how I’m going to do this problem.” (Dr. F)

Why is incubation important?

- Dr. G, from the focus group session: “When we are working on something, we are usually scribbling down on paper. When you go take a break,… you are thinking about it in your head without any visual aides….[walking around] forces me to think about it from a different point of view, and try different ways of thinking about it, often global, structural points of view.”
- Dr. F: “You just come back with a fresh mind. You’re zoomed in too much and you can’t see anything around it anymore.”
- Dr. A: “I do have a belief that if I walk away from something and come back it’s more likely that I’ll have an idea than if I just sit there.”

Discussion

- Educators want their students to have that "Eureka" or "AHA!" moment (Liljedahl, 2004)
- Incubation is important to mathematicians, so how can we show this effect to our students?
- One way might be to introduce "good" problems that require a good amount of thought.
- Schoenfeld (1982) described a "good" problem:
  - The problem needs to be accessible. That is, it is easily understood, and does not require specific knowledge to get into.
  - The problem can be approached from a number of different ways.
  - The problem should serve as an introduction to important mathematical ideas.
  - The problem should serve as a starting point for rich mathematical exploration and lead to more good problems.

Other Observations

- Two of the mathematicians misread the last theorem, “If S is a commutative semigroup, and K is a minimal ideal, then K is a group.”
  - Both emailed me after I had mentioned the misreading and sketched a proof in the email
- 3 professors looked for counter-examples of some theorems, noting that at first they seemed to be false claims

Future research

- Currently, I am in the first stages of finding the differences and similarities between other mathematics topics in order to better assist transition-to-proof courses.
- I would like to expand on how behavioral knowledge of logic-like structures helps to reduce the burden on working memory.
- I've had several professors ask me to code the proofs in a chapter of a textbook to see how much formal logic occurs in the proofs.

Future research (cont.)

- I would like to compare undergraduate and graduate students' data to that of the mathematicians in the proving process
- I would also like to use the data collection technique to help students' proving approaches
  - Akin to sports' "film sessions"
  - May also use a problem-solving framework so that students explicitly actions such as validation
  - May be helpful in transition-to-proof or other proof-based courses

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